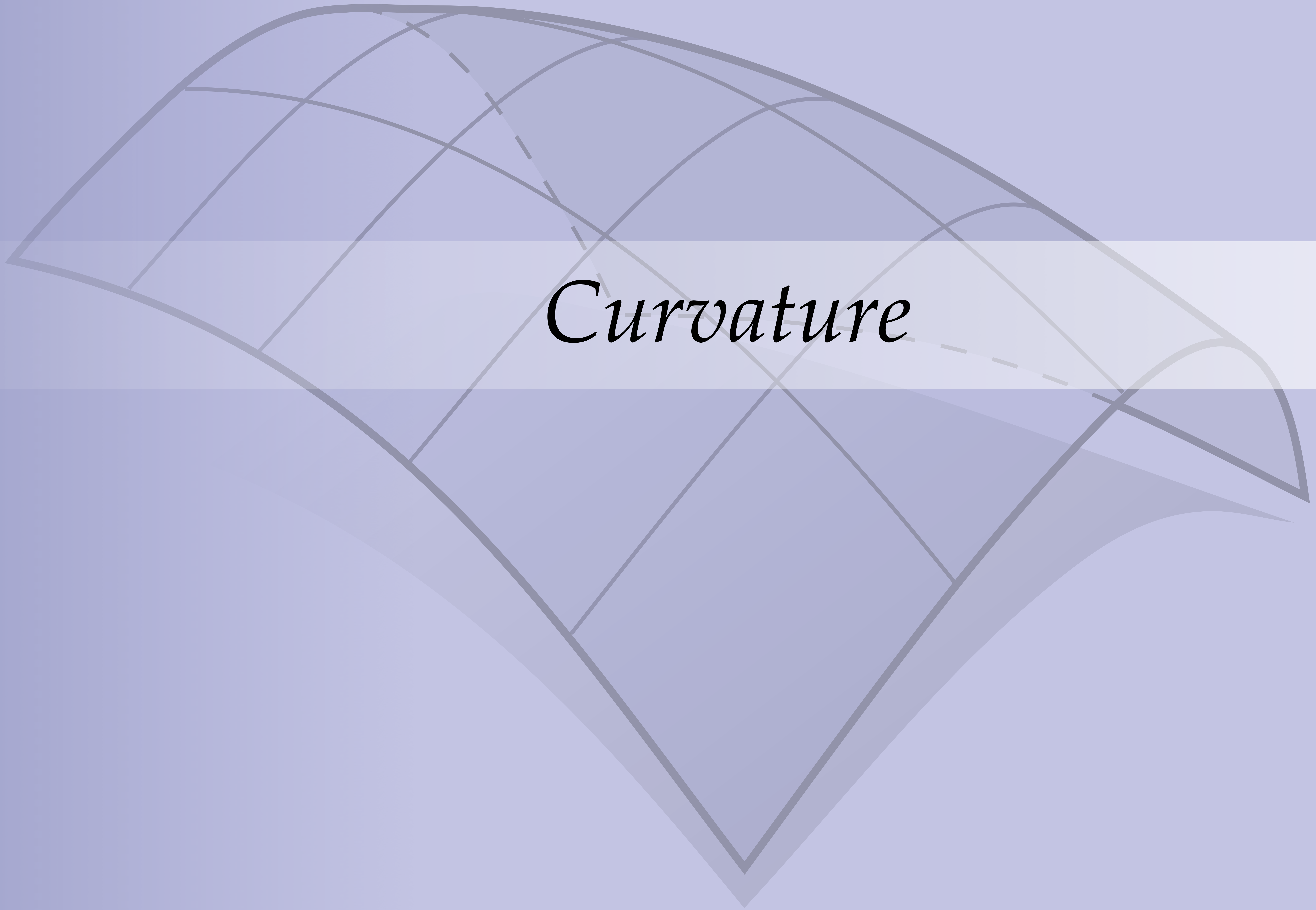
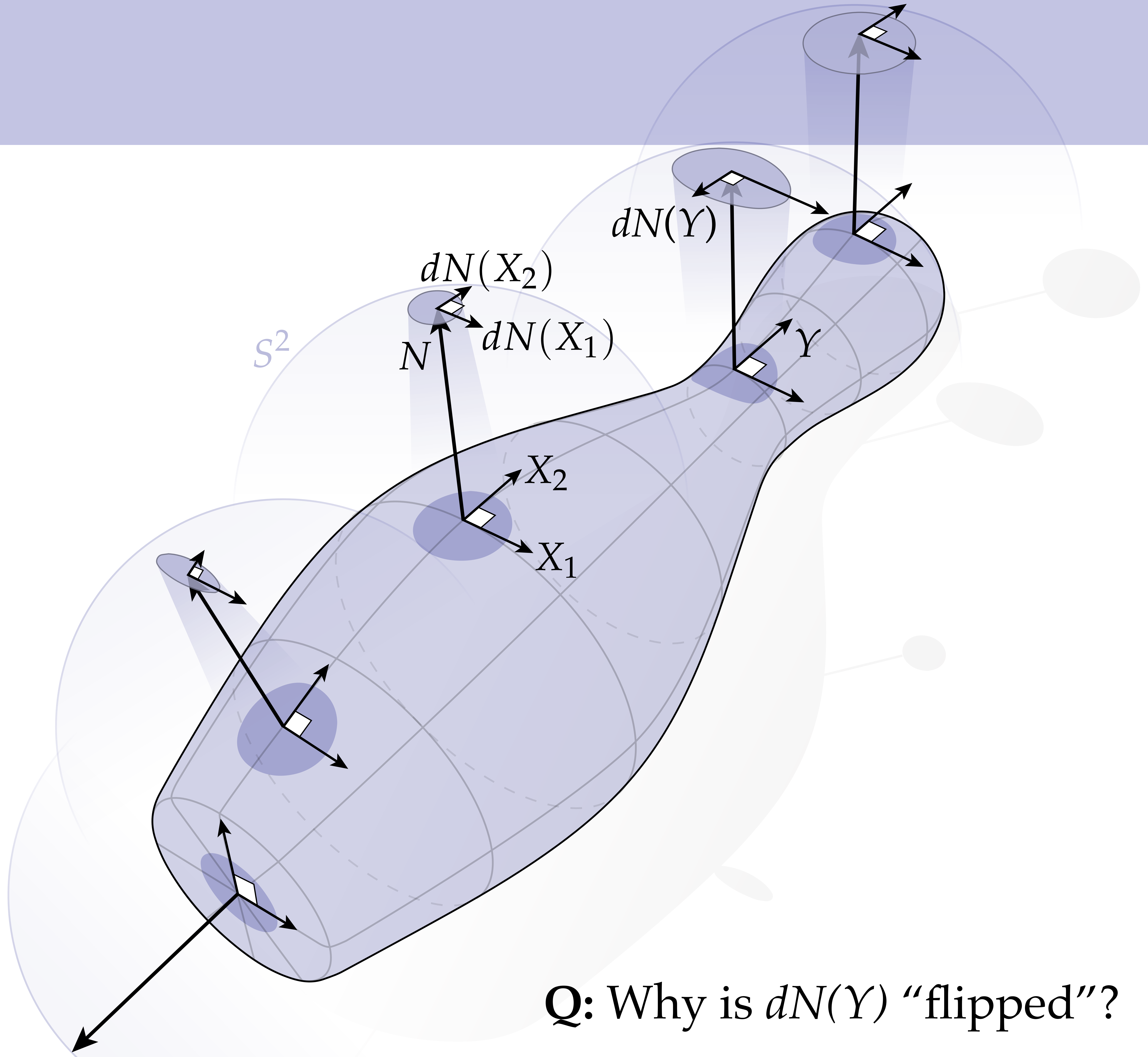
# *Curvature*



# *Weingarten Map*

- The **Weingarten map** $dN$ is the differential of the Gauss map $N$
- At each point, tells us the change in the normal vector along any given direction $X$
- Since change in *unit* normal cannot have any component in the normal direction, *dN(X)* is always tangent to the surface
- Can also think of it as a vector tangent to the unit sphere $S^2$




**Q:** Why is *dN(Y)* “flipped”?

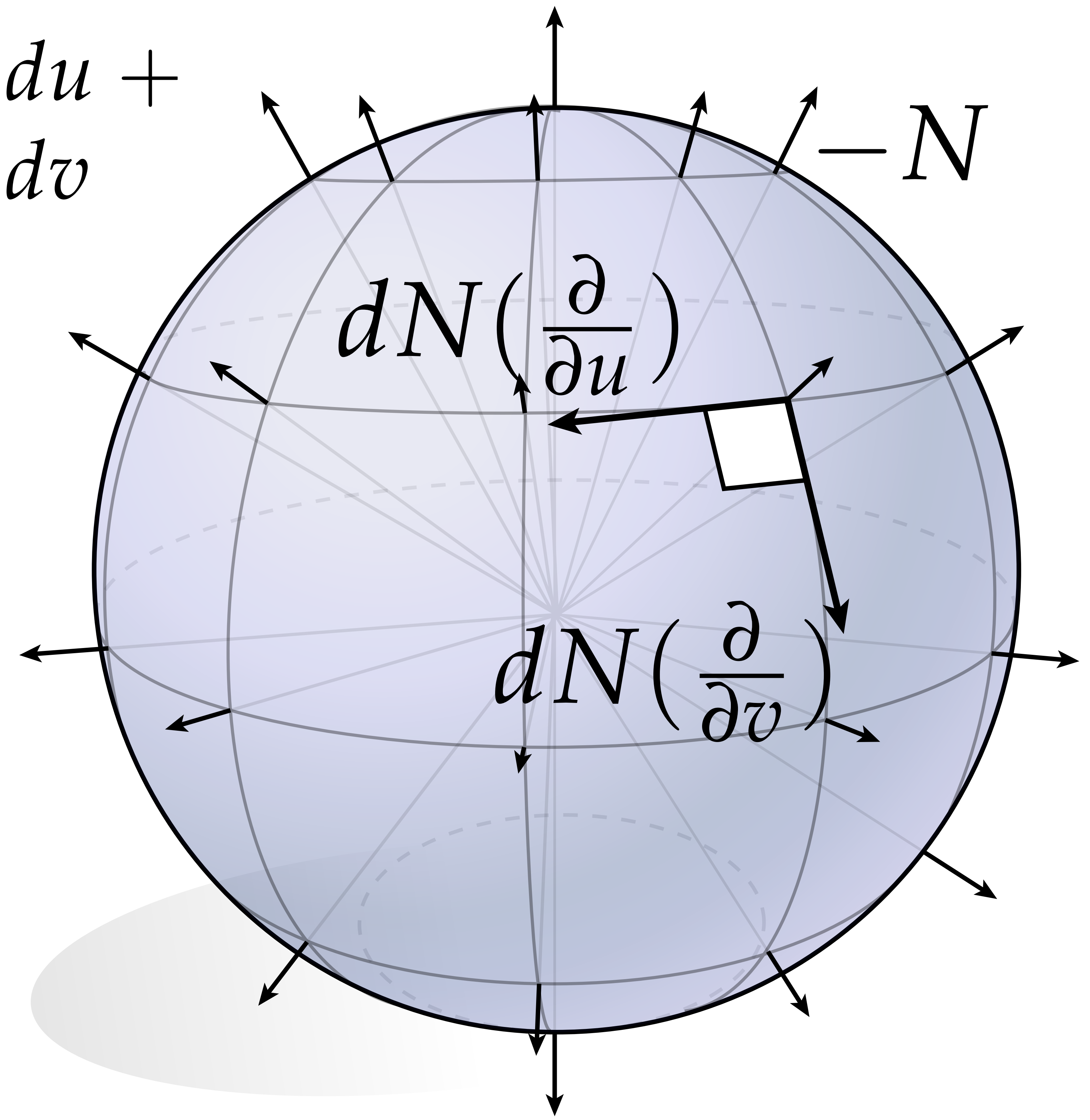
# *Weingarten Map—Example*

- Recall that for the sphere, $N = -f$. Hence, Weingarten map $dN$ is just $-df$:

$$f := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

$$df = \begin{array}{l} (\quad -\sin(u)\sin(v), \quad \cos(u)\sin(v), \quad 0 \quad)\ du\ + \\ (\quad \cos(u)\cos(v), \quad \cos(v)\sin(u), \quad -\sin(v) \quad)\ dv \end{array}$$

$$dN = \begin{array}{l} (\quad \sin(u)\sin(v), \ -\cos(u)\sin(v), \quad 0 \quad)\ du \\ (-\cos(u)\cos(v), \ -\cos(v)\sin(u), \ \sin(v) \ )\ dv \end{array}$$



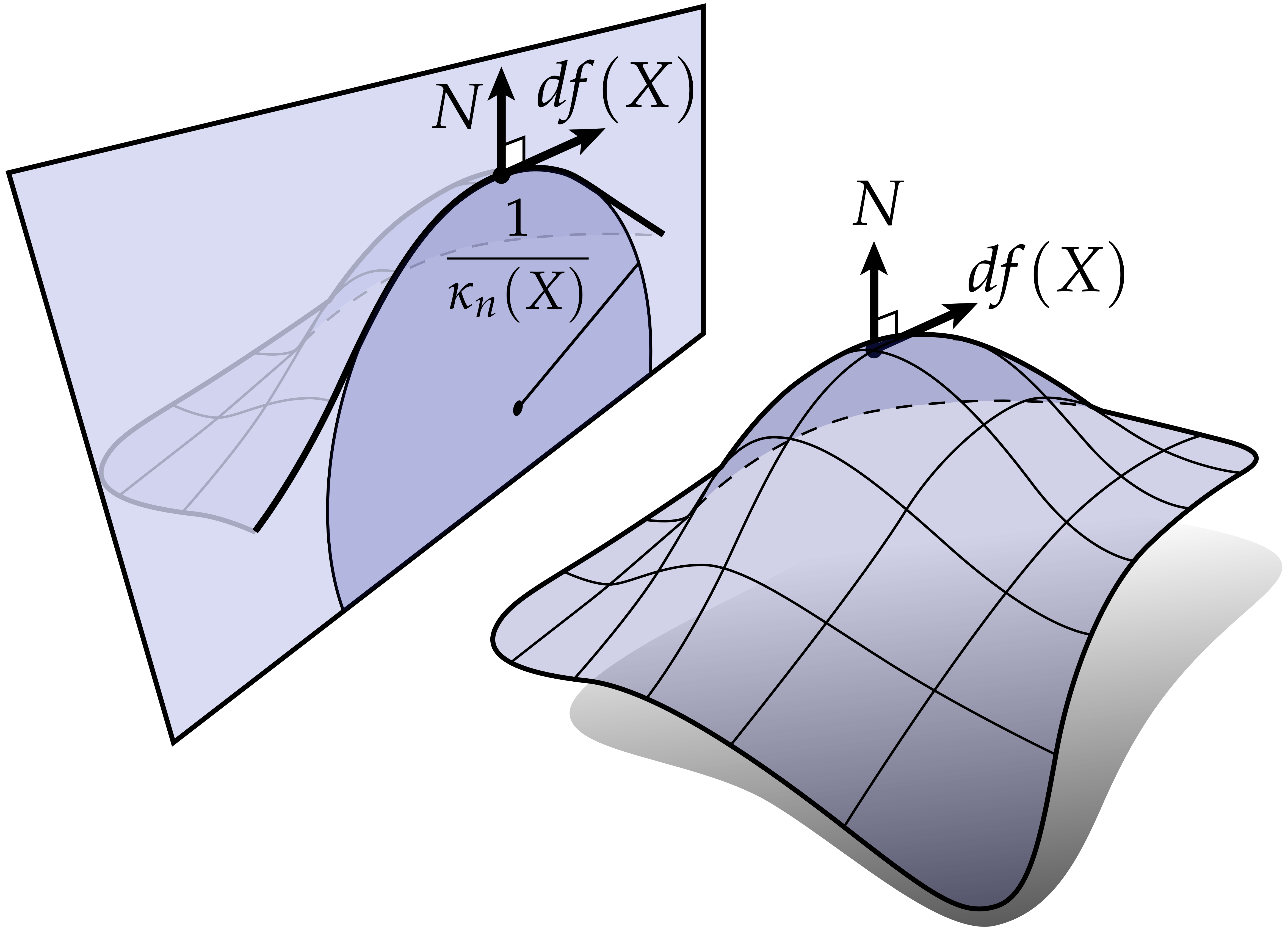

**Key idea:** computing the Weingarten map is no different from computing the differential of a surface.

# *Normal Curvature*

- For curves, curvature was the rate of change of the *tangent*; for immersed surfaces, we'll instead consider how quickly the *normal* is changing.*
- In particular, **normal curvature** is rate at which normal is bending along a given tangent direction:

$$\kappa_N(X) := \frac{\langle df(X), dN(X)\rangle}{|df(X)|^2}$$

- Equivalent to intersecting surface with normal-tangent plane and measuring the usual curvature of a plane curve




*For plane curves, what would happen if we instead considered change in $N$?

# *Normal Curvature—Example*

Consider a parameterized cylinder:

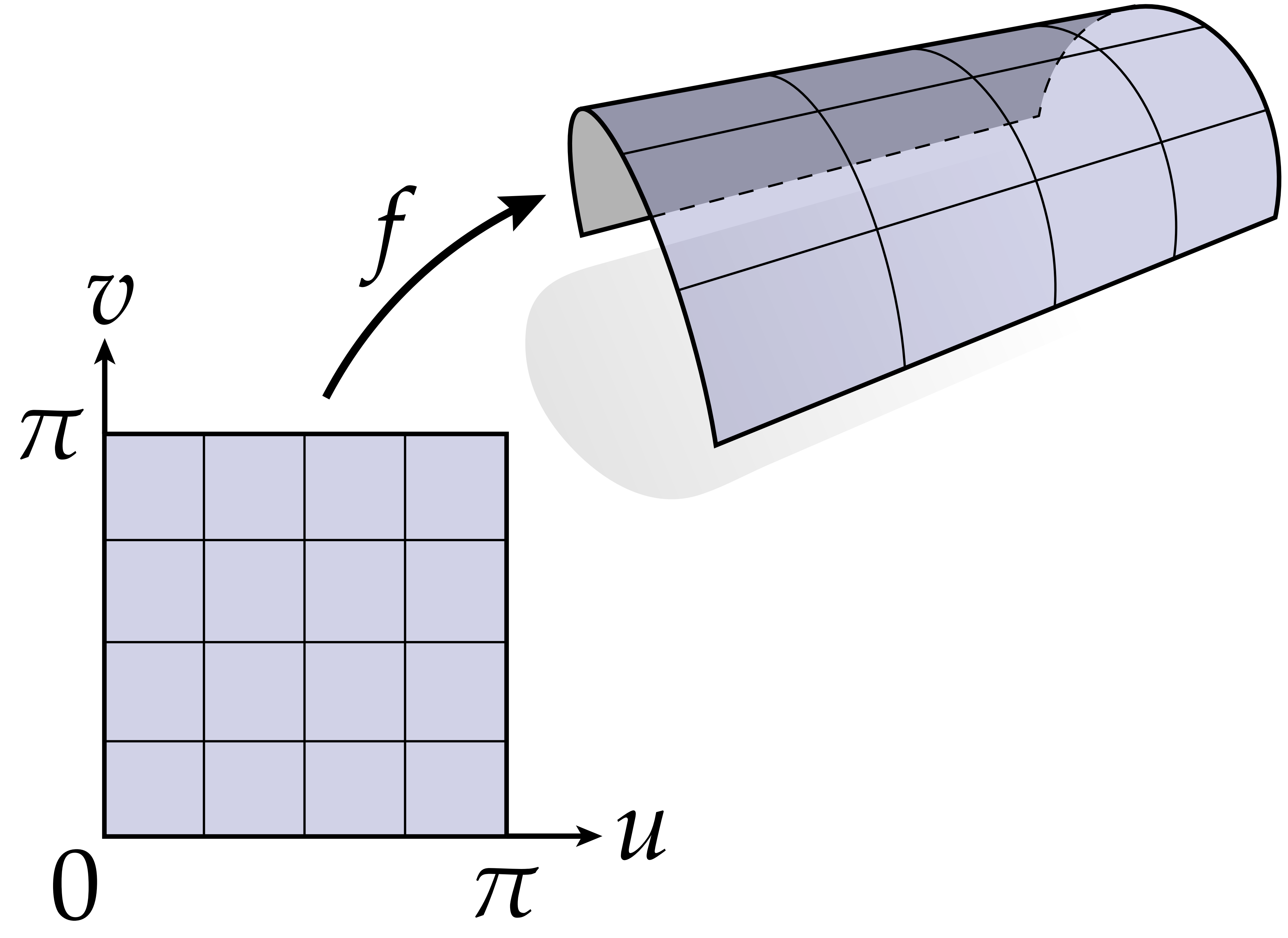
$$f(u,v) := (\cos(u), \sin(u), v)$$

$$df = (-\sin(u), \cos(u), 0)du + (0,0,1)dv$$

$$N = (-\sin(u), \cos(u), 0) \times (0,0,1)$$

$$= (\cos(u), \sin(u), 0)$$

$$dN = (-\sin(u), \cos(u), 0)du$$




$$\kappa_N(\tfrac{\partial}{\partial u}) = \frac{\langle df(\frac{\partial}{\partial u}), dN(\frac{\partial}{\partial u})\rangle}{|df(\frac{\partial}{\partial u})|^2} = \frac{(-\sin(u),\cos(u),0)\cdot(-\sin(u),\cos(u),0)}{|(-\sin(u),\cos(u),0)|^2} = 1$$

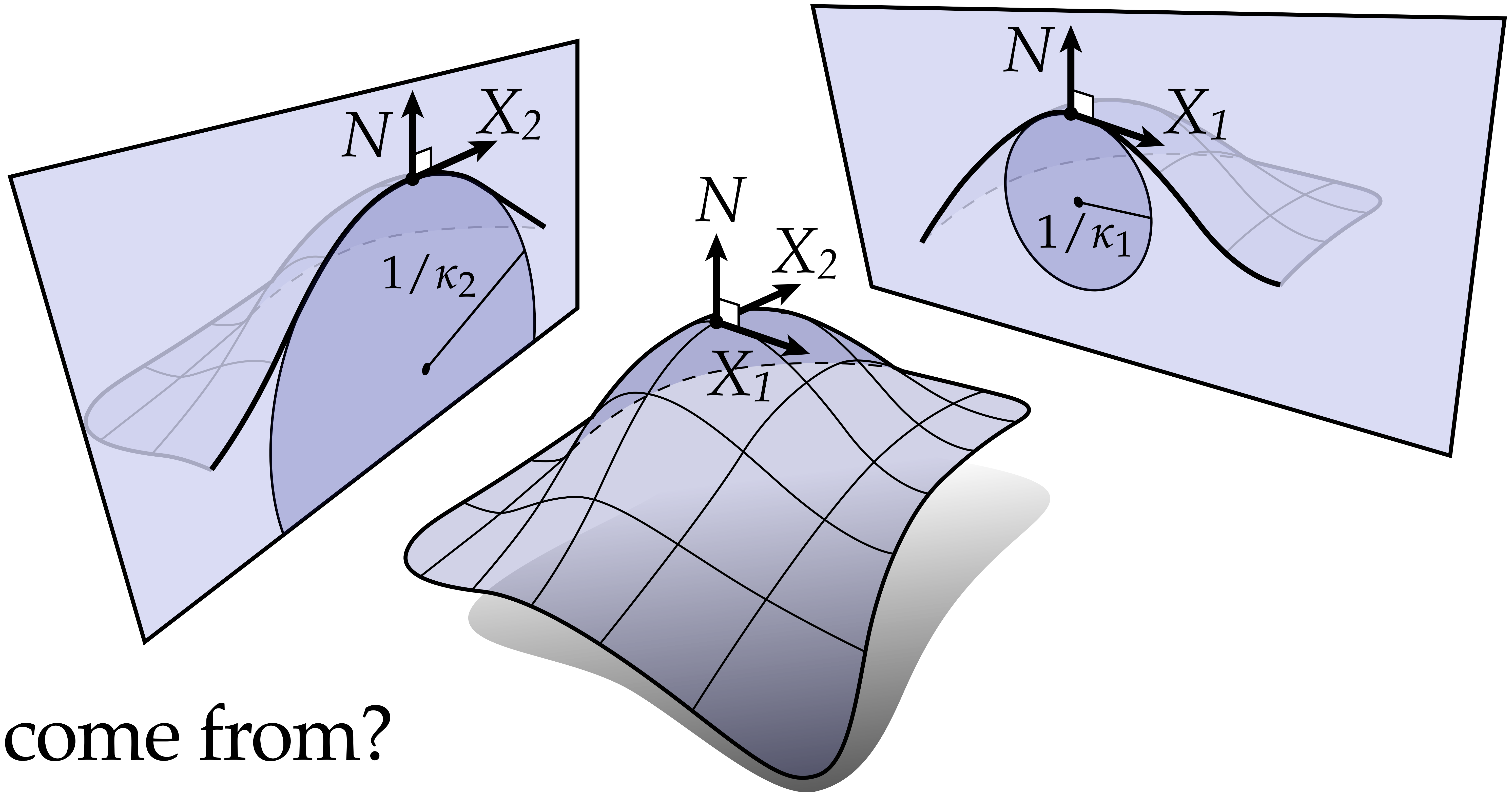
$$\kappa_N(\tfrac{\partial}{\partial v}) = \cdots = 0$$

**Q:** Does this result make sense geometrically?

# *Principal Curvature*

- Among all directions $X$, there are two **principal directions** $X_1$, $X_2$ where normal curvature has minimum/maximum value (respectively)
- Corresponding normal curvatures are the **principal curvatures**
- Two critical facts*:

  1. $g(X_1, X_2) = 0$
  2. $dN(X_i) = \kappa_i df(X_i)$




Where do these relationships come from?

# *Shape Operator*

- The change in the normal $N$ is always *tangent* to the surface
- Must therefore be some linear map $S$ from tangent vectors to tangent vectors, called the **shape operator**, such that

$$df(SX) = dN(X)$$

- Principal directions are the *eigenvectors* of S
- Principal curvatures are *eigenvalues* of S
- **Note:** $S$ is not a symmetric matrix! Hence, eigenvectors are not orthogonal in $R^2$; only orthogonal with respect to induced metric $g$.

# *Shape Operator — Example*

Consider a nonstandard parameterization of the cylinder (*sheared* along z):

$$f(u,v) := (\cos(u), \sin(u), u+v) \qquad df = (-\sin(u), \cos(u), 1)du + (0,0,1)dv$$

$$N = (\cos(u), \sin(u), 0) \qquad dN = (-\sin(u), \cos(u), 0)du$$

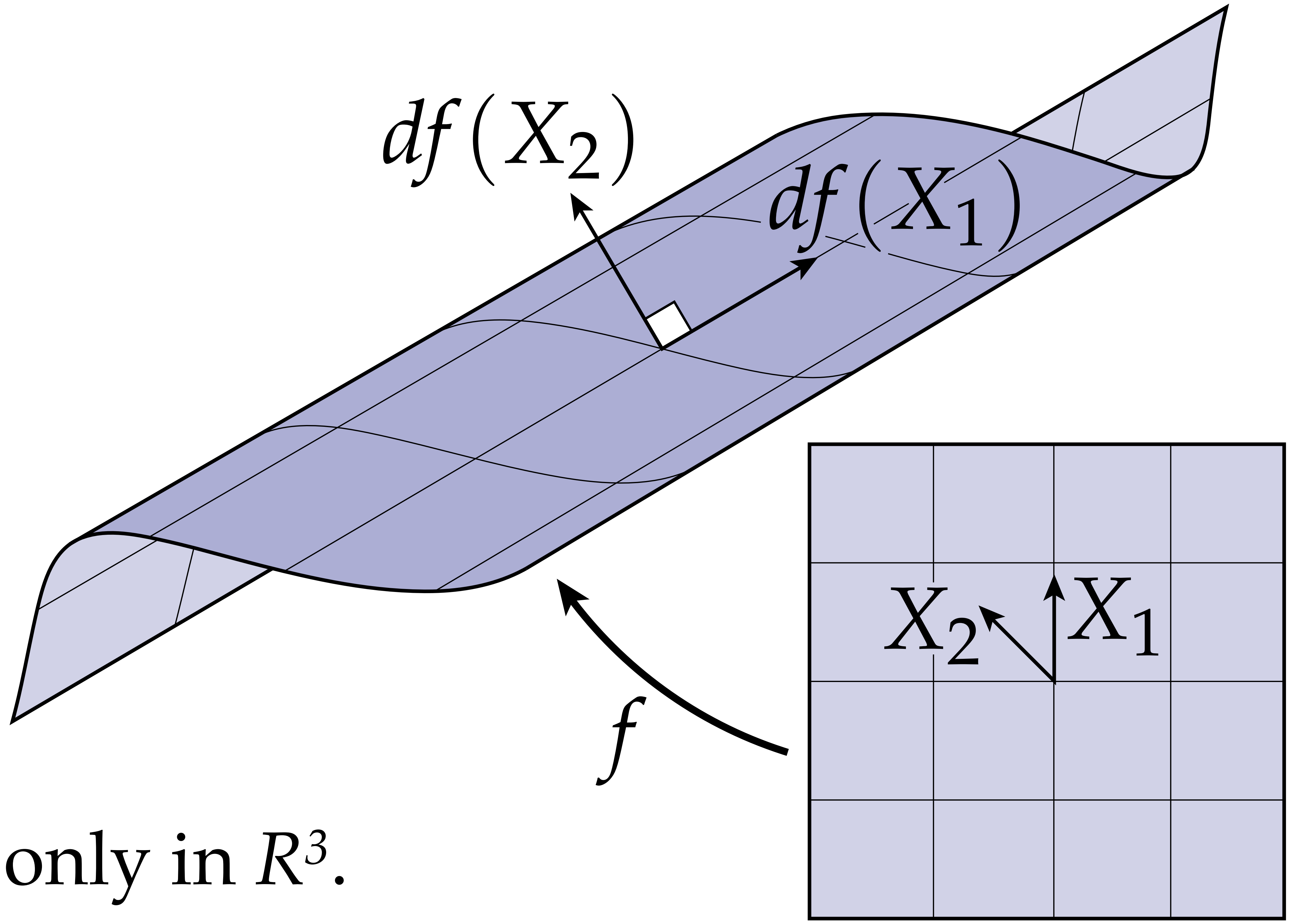
$$\boxed{df \circ S = dN}$$

$$\begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \qquad X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$df(X_1) = (0,0,1) \qquad \kappa_1 = 0$$

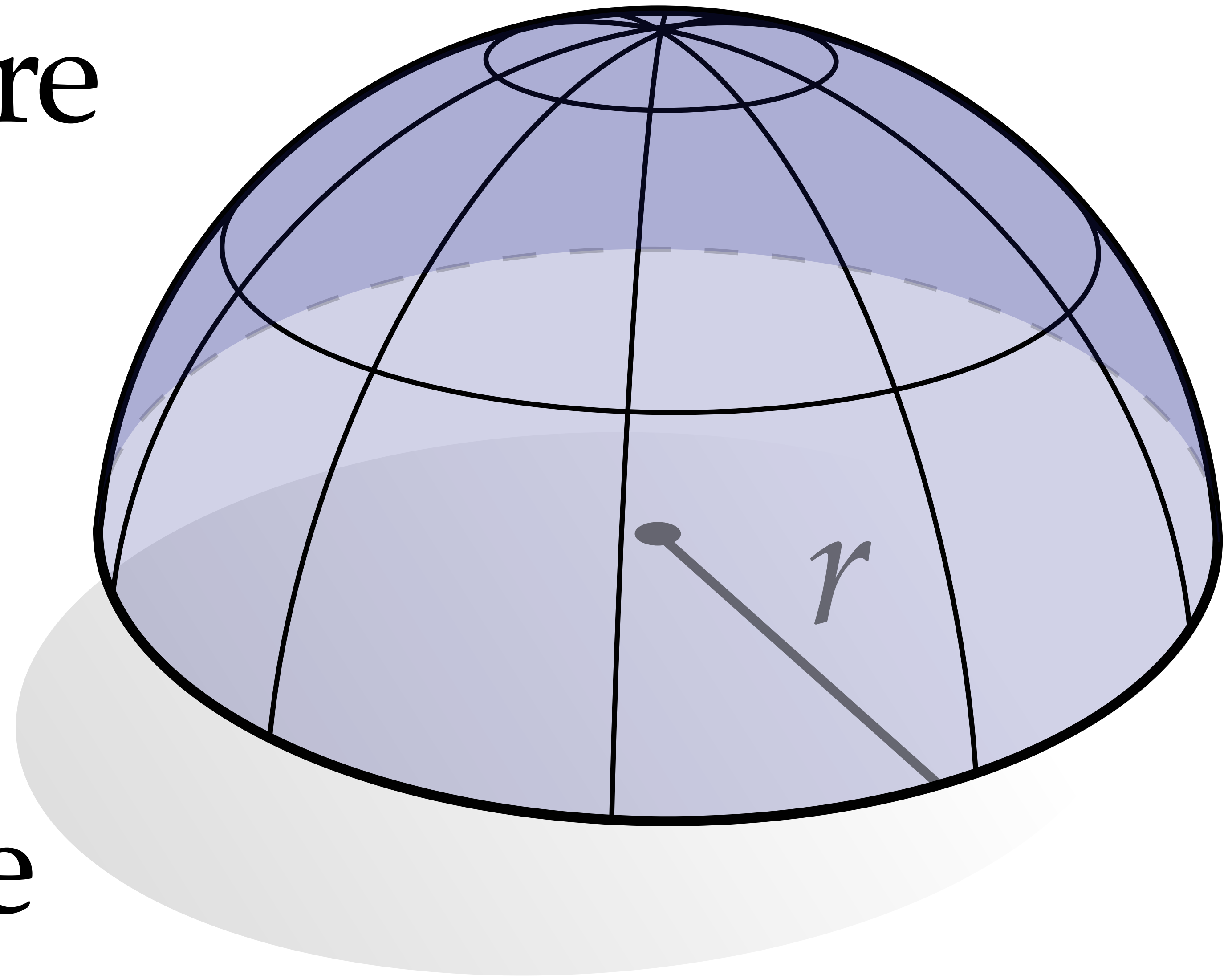
$$df(X_2) = (\sin(u), -\cos(u), 0) \qquad \kappa_2 = 1$$




**Key observation:** principal directions orthogonal only in $R^3$.

# *Umbilic Points*

- Points where principal curvatures are equal are called **umbilic points**
- Principal *directions* are not uniquely determined here
- What happens to the shape operator $S$?
  - May still have full rank!
  - Just have repeated eigenvalues, 2-dim. eigenspace

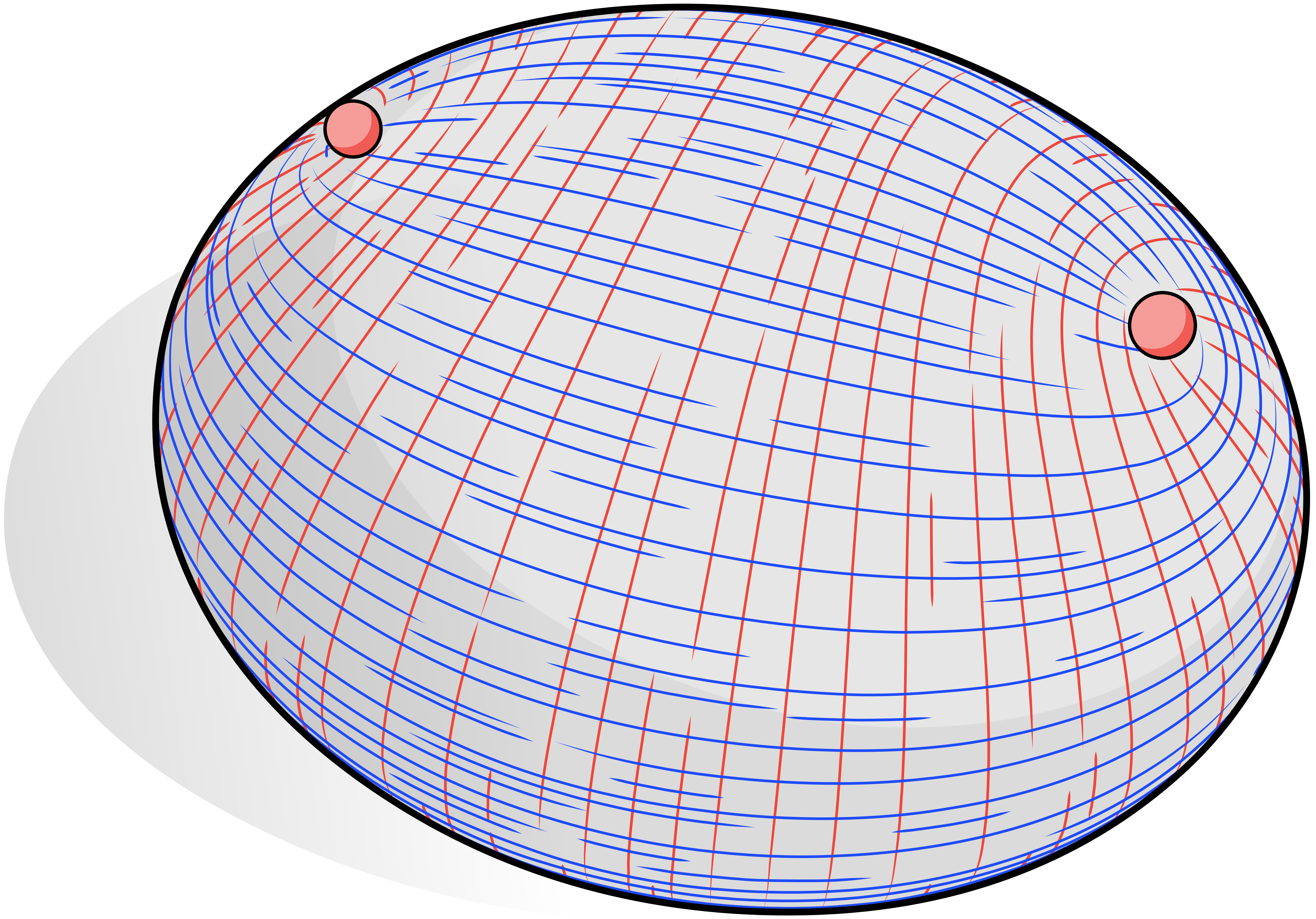


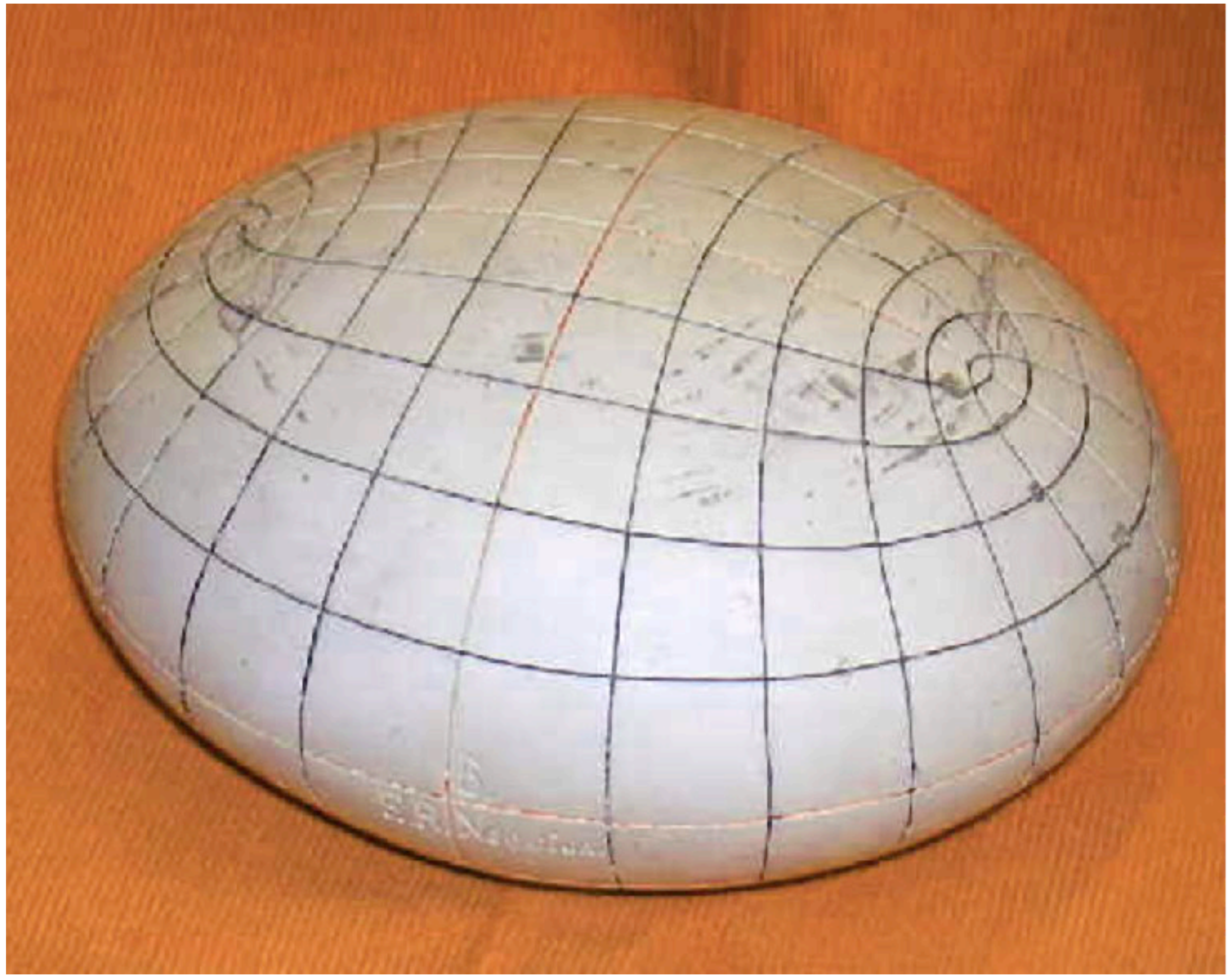

$$S = \begin{bmatrix} 1/r & 0 \\ 0 & 1/r \end{bmatrix} \qquad \kappa_1 = \kappa_2 = \frac{1}{r} \qquad \forall X,\ SX = \tfrac{1}{r}X$$

Could still of course choose (arbitrarily) an orthonormal pair $X_1$, $X_2$…

# *Principal Curvature Nets*

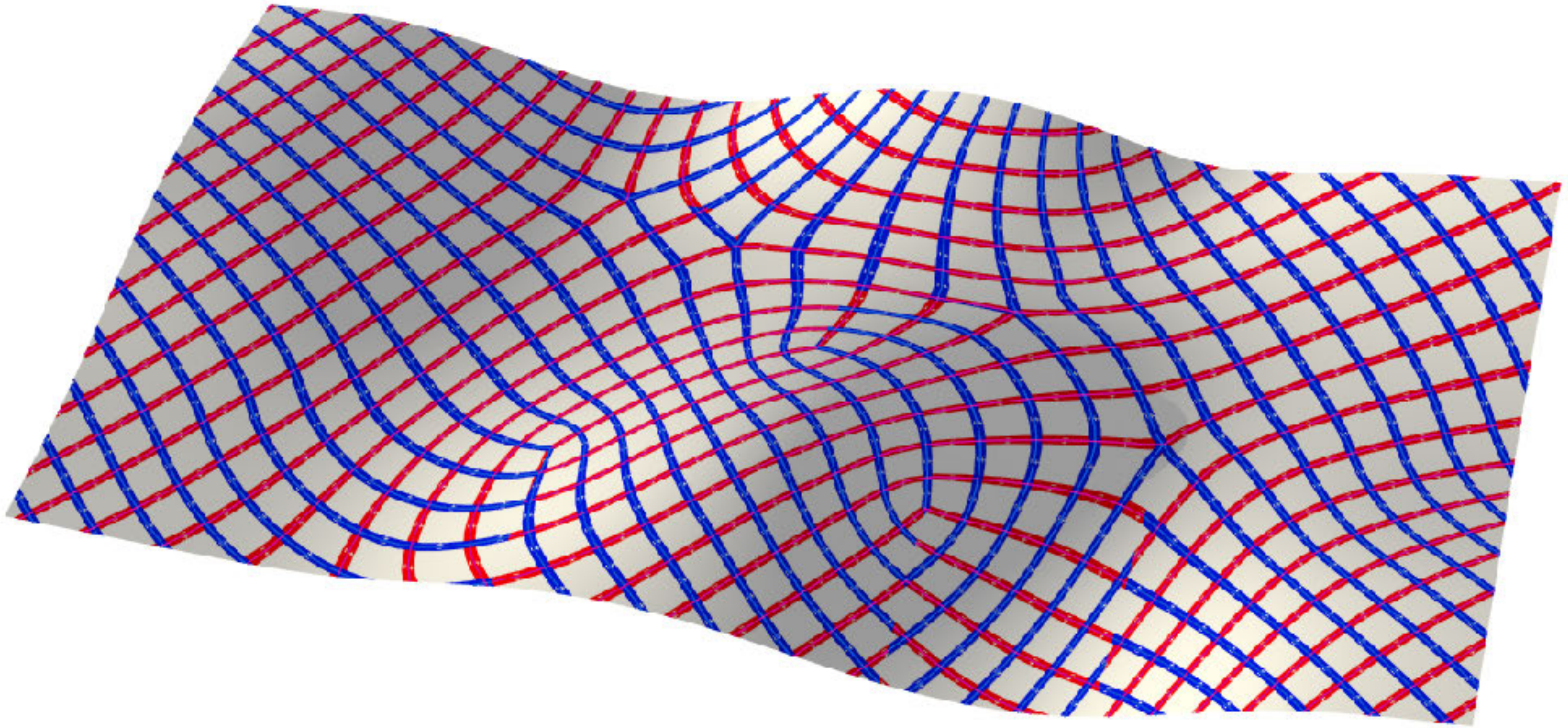
- Walking along principal direction field yields **principal curvature lines**
- Collection of all such lines is called the **principal curvature network**

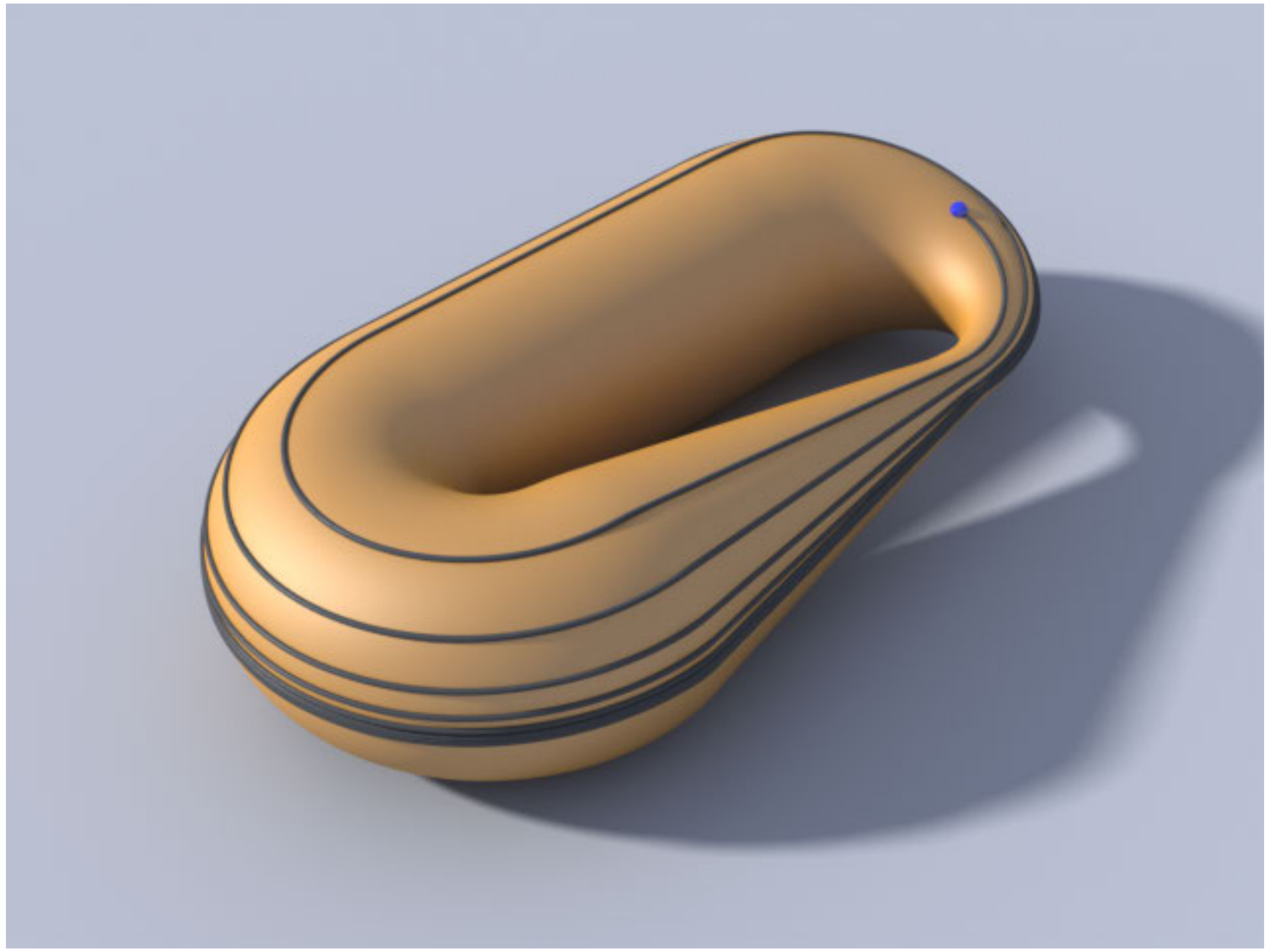




# *Separatrices and Spirals*

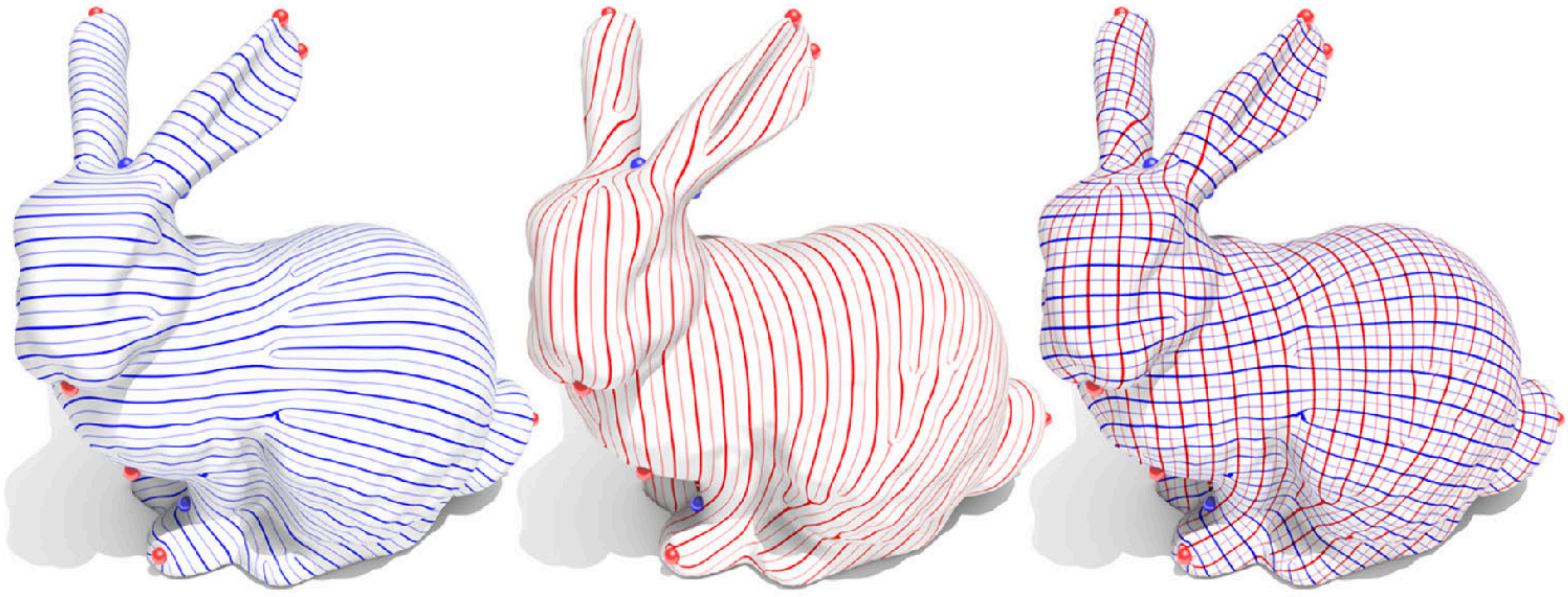
- If we walk along a principal curvature line, where do we end up?
- Sometimes, a curvature line terminates at an umbilic point in both directions; these so-called **separatrices** (can) split network into regular patches.
- Other times, we make a closed loop. More often, however, behavior is *not* so nice!





# *Application—Quad Remeshing*

- Recent approach to meshing: construct net *roughly* aligned with principal curvature—but with separatrices & loops, not spirals.



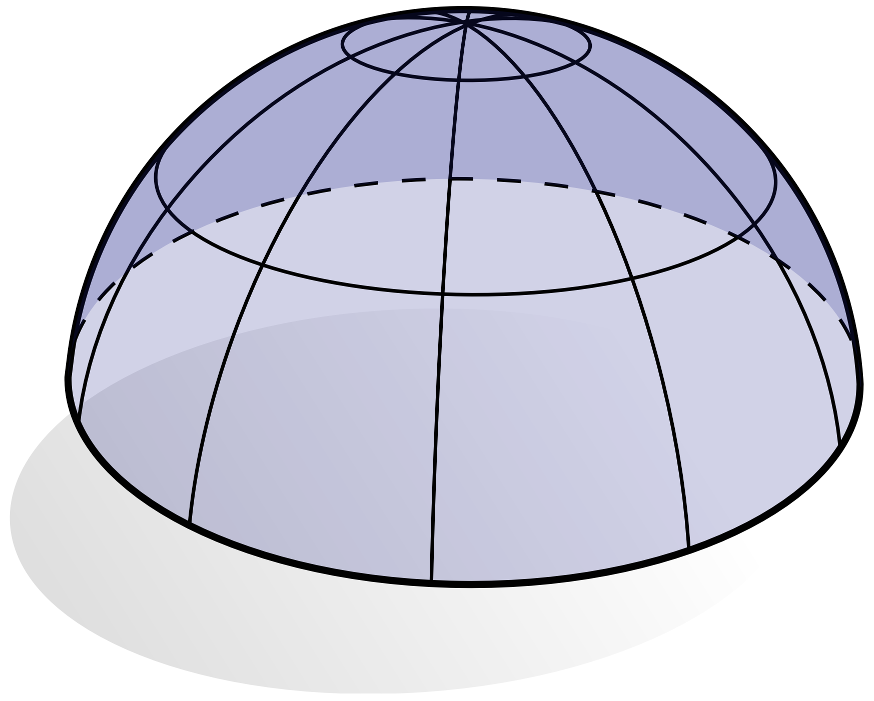
from Knöppel, Crane, Pinkall, Schröder, *"Stripe Patterns on Surfaces"*

# *Gaussian and Mean Curvature*

*Gaussian* and *mean* curvature also fully describe local bending:

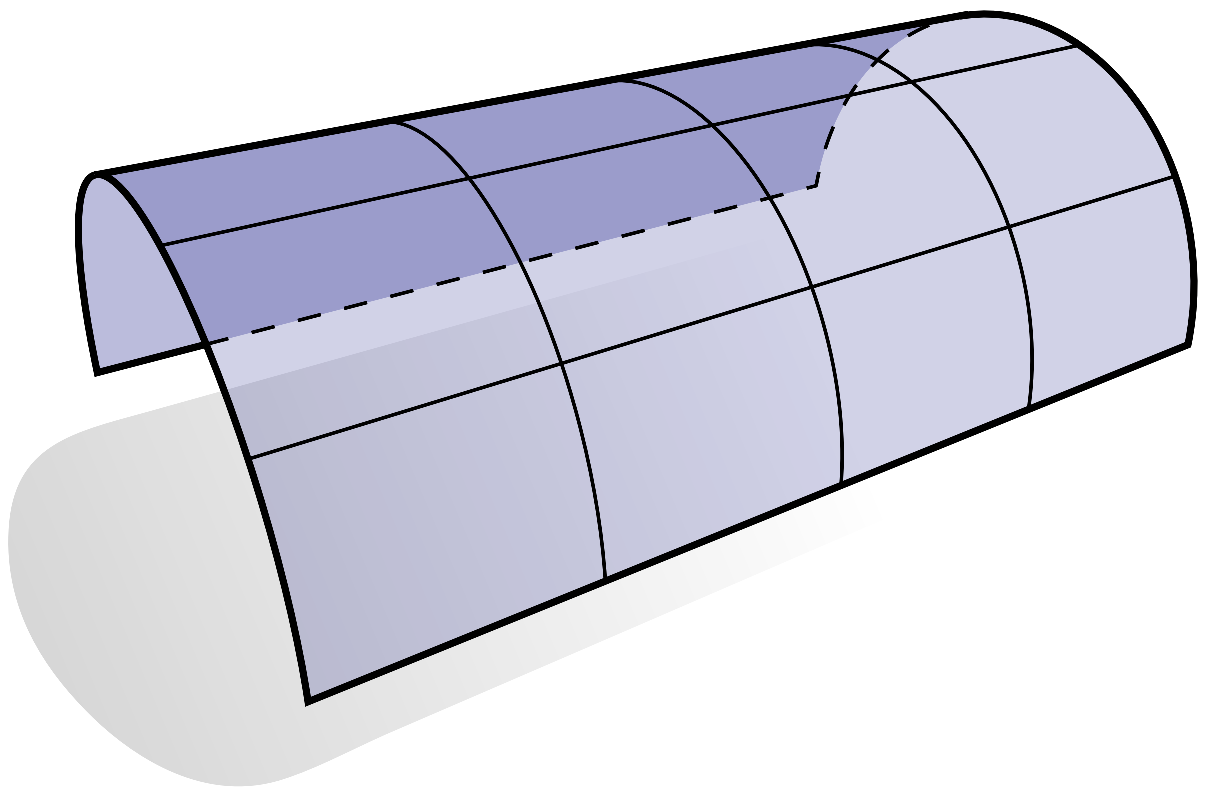
$$\textbf{Gaussian} \quad K := \kappa_1\kappa_2$$

$$\textbf{mean*} \quad H := \tfrac{1}{2}(\kappa_1 + \kappa_2)$$



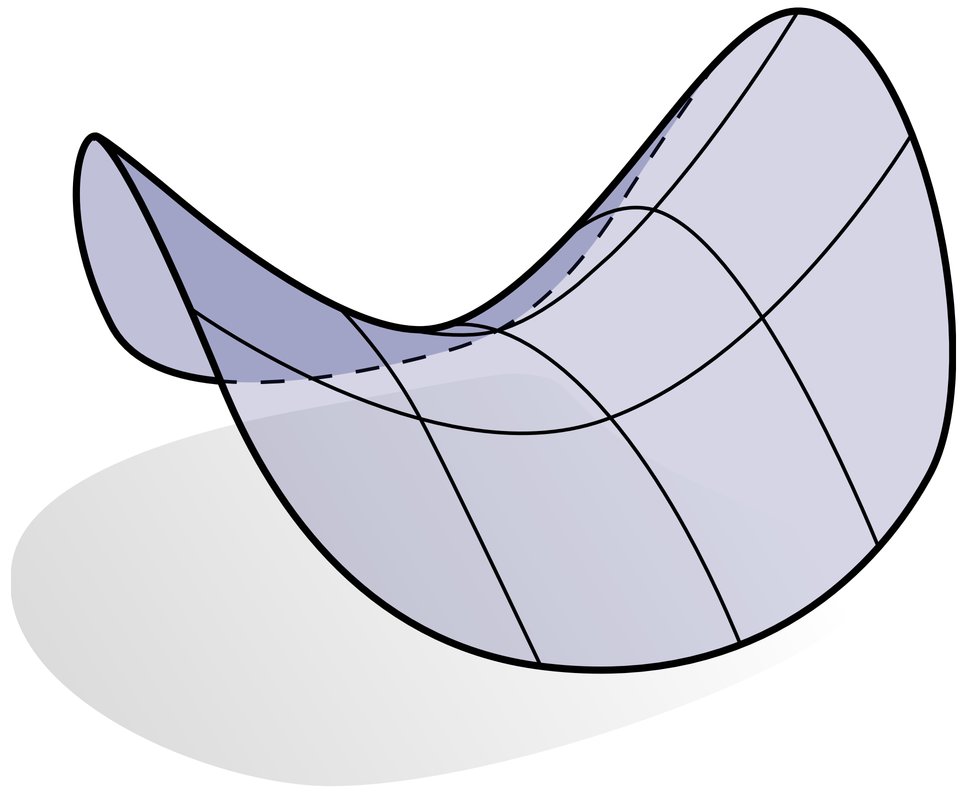
$K > 0$

$H \neq 0$



*"developable"* $\boxed{K = 0}$

$H \neq 0$



$K < 0$

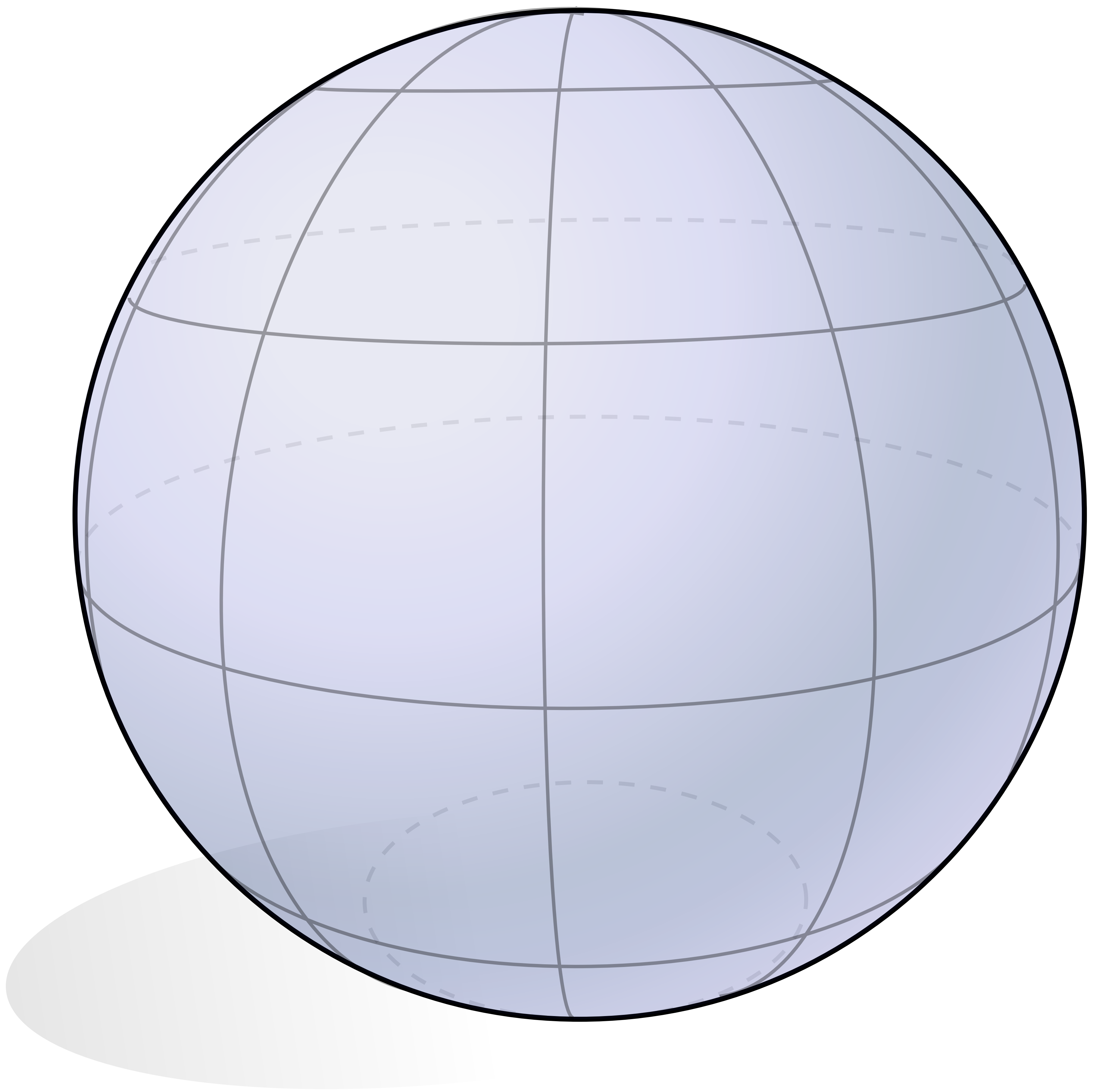
*"minimal"* $\boxed{H = 0}$

*Warning:** another common convention is to omit the factor of $1/2$

# *Total Mean Curvature?*

**Theorem** (Minkowski): for a regular closed embedded surface,

$$\int_M H\, dA \geq \sqrt{4\pi A}$$

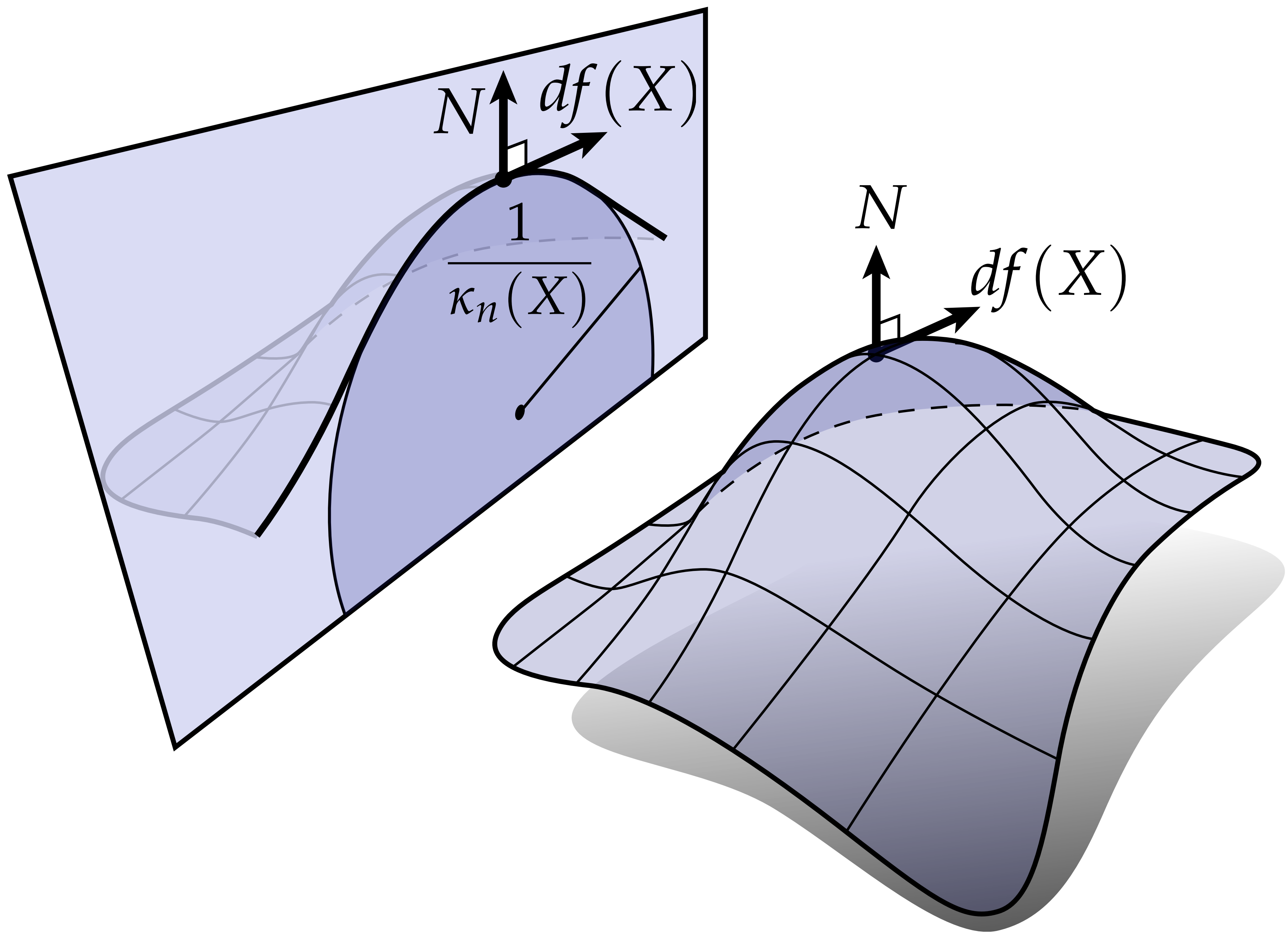


**Q:** When do we get equality?

**A:** For a sphere.

# *Second Fundamental Form*

- Second fundamental form is closely related to principal curvature
- Can also be viewed as change in *first* fundamental form under motion in normal direction
- Why "fundamental?"First & second fundamental forms play role in important theorem…




$$\mathbf{II}(X,Y) := \langle dN(X), df(Y)\rangle$$

$$\kappa_N(X) := \frac{df(X), dN(X)}{|df(X)|^2} = \frac{\mathbf{II}(X,X)}{\mathbf{I}(X,X)}$$

# *Fundamental Theorem of Surfaces*

- **Fact.** Two surfaces in $R^3$ are congruent if and only if they have the same first and second fundamental forms
  - …However, not every pair of bilinear forms **I**, **II** on a domain $U$ describes a valid surface—must satisfy the **Gauss Codazzi** equations
- Analogous to fundamental theorem of plane curves: determined up to rigid motion by curvature
  - …However, for *closed* curves not every curvature function is valid (*e.g.*, must integrate to $2k\pi$)