Length Shortening Flow

• The objective for length shortening flow is simply the total length of the curve; the flow is then the \((L^2)\) gradient flow.

• For closed curves, several interesting features (Gage-Grayson-Hamilton):
  • Center of mass is preserved
  • Curves flow to “round points”
  • Embedded curves remain embedded

\[
\text{length}(\gamma) := \int_0^L \left| \frac{d}{ds} \gamma \right| \, ds
\]

\[
\frac{d}{dt} \gamma = -\nabla_\gamma \text{length}(\gamma)
\]

credit: Sigurd Angenent
Length Shortening Flow

Let \( \text{length}(\gamma) \) denote the total length of a regular plane curve \( \gamma : [0, L] \rightarrow \mathbb{R}^2 \), and consider a variation \( \eta : [0, L] \rightarrow \mathbb{R}^2 \) vanishing at endpoints. One can then show that

\[
\frac{d}{d\varepsilon} |_{\varepsilon=0} \text{length}(\gamma + \varepsilon \eta) = - \int_0^L \langle \eta(s), \kappa(s) N(s) \rangle \, ds
\]

**Key idea:** quickest way to reduce length is to move in the direction \( \kappa N \).
Length Shortening Flow—Forward Euler

• At each moment in time, move curve in normal direction with speed proportional to curvature

• “Smoothes out” curve (e.g., noise), eventually becoming circular

• Discretize by replacing time derivative with difference in time; smooth curvature with one (of many) curvatures

• Repeatedly add a little bit of $\kappa N$ ("forward Euler method")

\[
\frac{d}{dt} \gamma(s, t) = -\kappa(s, t)N(s, t)
\]

\[
\frac{\gamma_i^{t+1} - \gamma_i^t}{\tau} = -\kappa_i^t N_i^t
\]

\[
\Rightarrow \gamma_i^{t+1} = \gamma_i^t - \tau \kappa_i^t N_i^t
\]

smooth discrete
Elastic Flow

- Basic idea: rather than shrinking length, try to reduce bending (curvature)
- Objective is integral of squared curvature; elastic flow is then gradient flow on this objective
- Minimizers are called elastic curves
- More interesting w/ constraints (e.g., endpoint positions & a tangents)

\[ E(\gamma) := \int_0^L \kappa(s)^2 \, ds \]
\[ \frac{d}{dt} \gamma = -\nabla_\gamma E(\gamma) \]
Isometric Elastic Flow

• Different way to smooth out a curve is to directly “shrink” curvature

• Discrete case: “scale down” turning angles, then use the fundamental theorem of discrete plane curves to reconstruct

• Extremely stable numerically; exactly preserves edge lengths

• Challenge: how do we make sure closed curves remain closed?

From Crane et al, “Robust Fairing via Conformal Curvature Flow”
Elastic Rods

• For space curve, can also try to minimize both curvature and torsion

• Both in some sense measure “non-straightness” of curve

• Provides rich model of elastic rods

• Lots of interesting applications (simulating hair, laying cable, …)

From Bergou et al, “Discrete Elastic Rods”
• Readings from papers on curve algorithms (will be posted online)
From Curves to Surfaces

• Previously: saw how to talk about 1D curves (both smooth and discrete)

• Today: will study 2D curved surfaces (both smooth and discrete)

• Some concepts remain the same (e.g., differential); others need to be generalized (e.g., curvature)

• Still use exterior calculus as our lingua franca
Surfaces—Local vs. Global View

• So far, we’ve only studied exterior calculus in $R^n$

• Will therefore be easiest to think of surfaces expressed in terms of patches of the plane (local picture)

• Later, when we study topology & smooth manifolds, we’ll be able to more easily think about “whole surfaces” all at once (global picture)

• Global picture is much better model for discrete surfaces (meshes)…
Parameterized Surfaces
A parameterized surface is a map from a two-dimensional region $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$:

$$f : U \rightarrow \mathbb{R}^n$$

The set of points $f(U)$ is called the image of the parameterization.
Parameterized Surface—Example

As an example, we can express a saddle as a parameterized surface:

\[ U := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \]

\[ f : U \rightarrow \mathbb{R}^3; (u, v) \mapsto (u, v, u^2 - v^2) \]
Reparameterization

• Many different parameterized surfaces can have the same image:

\[ U := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \]

\[ f : U \rightarrow \mathbb{R}^3; (u, v) \mapsto (u + v, u - v, 4uv) \]

This “reparameterization symmetry” can be a major challenge in applications—e.g., trying to decide if two parameterized surfaces (or meshes) describe the same shape.

Analogy: graph isomorphism
Embedded Surface

- Roughly speaking, an **embedded** surface does not self-intersect.
- More precisely, a parameterized surface is an embedding if it is a continuous injective map, and has a continuous inverse on its image.
Differential of a Surface

Intuitively, the differential of a parameterized surface tells us how tangent vectors on the domain get mapped to vectors in space:

We say that \( df \) “pushes forward” vectors \( X \) into \( \mathbb{R}^n \), yielding vectors \( df(X) \).
Differential in Coordinates

In coordinates, the differential is simply the exterior derivative:

\[ f : U \to \mathbb{R}^3; \quad (u, v) \mapsto (u, v, u^2 - v^2) \]

\[
\begin{align*}
    df &= \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv = \\
    &= (1, 0, 2u) du + (0, 1, -2v) dv
\end{align*}
\]

Pushforward of a vector field:

\[ X := \frac{3}{4} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \]

\[ df(X) = \frac{3}{4} (1, -1, 2(u + v)) \]

E.g., at \( u=v=0 \):

\( \left( \frac{3}{4}, -\frac{3}{4}, 0 \right) \)
Definition. Consider a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, and let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. Then the Jacobian of $f$ is the matrix

$$J_f := \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{bmatrix},$$

where $f^1, \ldots, f^m$ are the components of $f$ w.r.t. some coordinate system on $\mathbb{R}^m$. This matrix represents the differential in the sense that $df(X) = J_f X$.

(In solid mechanics, also known as the deformation gradient.)

Note: does not generalize to infinite dimensions! (E.g., maps between functions.)
Immersed Surface

• A parameterized surface $f$ is an *immersion* if its differential is nondegenerate, i.e., if $df(X) = 0$ if and only if $X = 0$.

**Intuition:** no region of the surface gets “pinched”
Consider the standard parameterization of the sphere:

\[ f(u, v) := (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v)) \]

\[
\begin{align*}
df &= \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv = \\
&= \left( -\sin(u) \sin(v), \cos(u) \sin(v), 0 \right) du + \\
&\quad \left( \cos(u) \cos(v), \cos(v) \sin(u), -\sin(v) \right) dv
\end{align*}
\]

**Q:** Is \( f \) an immersion?

**A:** No: when \( v = 0 \) we get

\[
\left( 0, 0, 0 \right) du + \left( \cos(u), \sin(u), -\sin(v) \right) dv
\]

*Nonzero tangents mapped to zero!*
Immersion vs. Embedding

- In practice, ensuring that a surface is globally embedded can be challenging.

- Immersions are typically “nice enough” to define local quantities like tangents, normals, metric, etc.

- Immersions are also a natural model for the way we typically think about meshes: most quantities (angles, lengths, etc.) are perfectly well-defined, even if there happen to be self-intersections.
Sphere Eversion

Turning a Sphere Inside-Out (1994)

https://youtu.be/-6g3ZcmJ7k
Riemannian Metric
Riemann Metric

• Many quantities on manifolds (curves, surfaces, etc.) ultimately boil down to measurements of lengths and angles of tangent vectors.

• This information is encoded by the so-called Riemannian metric.*

• Abstractly: smoothly-varying positive-definite bilinear form.

• For immersed surface, can (and will!) describe more concretely / geometrically.

*Note: not the same as a point-to-point distance metric $d(x,y)$. 
Metric Induced by an Immersion

• Given an immersed surface $f$, how should we measure inner product of vectors $X$, $Y$ on its domain $U$?

• We should not use the usual inner product on the plane! (Why not?)

• Planar inner product tells us nothing about actual length & angle on the surface (and changes depending on choice of parameterization!)

• Instead, use induced metric

$$g(X, Y) := \langle df(X), df(Y) \rangle$$

Key idea: must account for “stretching”
Induced Metric—Matrix Representation

• Metric is a bilinear map from a pair of vectors to a scalar, which we can represent as a 2x2 matrix $I$ called the first fundamental form:

$$g(X, Y) = X^T I Y$$

$$\Rightarrow I_{ij} = g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \left\langle df \left( \frac{\partial}{\partial x^i} \right), df \left( \frac{\partial}{\partial x^j} \right) \right\rangle$$

• Alternatively, can express first fundamental form via Jacobian:

$$g(X, Y) = \left\langle df(X), df(Y) \right\rangle = (J_f X)^T (J_f Y) = X^T (J_f^T J_f) Y$$

$$\Rightarrow I = J_f^T J_f$$
Induced Metric — Example

Can use the differential to obtain the induced metric:

\[ f : U \rightarrow \mathbb{R}^3; (u, v) \mapsto (u, v, u^2 - v^2) \]

\[ df = (1, 0, 2u)du + (0, 1, -2v)dv \]

\[ J_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix} \]

\[ I = J_f^T J_f \]

\[ = \begin{bmatrix} 1 + 4u^2 & -4uv \\ -4uv & 1 + 4v^2 \end{bmatrix} \]
Conformal Coordinates

• As we’ve just seen, there can be a complicated relationship between length & angle on the domain (2D) and the image (3D)

• For curves, we simplified life by using an arc-length or isometric parameterization: lengths on domain are identical to lengths along curve

• For surfaces, usually not possible to preserve all lengths (e.g., globe). Remarkably, however, can always preserve angles (conformal)

• Equivalently, a parameterized surface is conformal if at each point the induced metric is simply a positive rescaling of the 2D Euclidean metric
Example (Enneper Surface)

Consider the surface

\[ f(u, v) := \begin{bmatrix} uv^2 + u - \frac{1}{3}u^3 \\ \frac{1}{3}v(v^2 - 3u^2 - 3) \\ (u - v)(u + v) \end{bmatrix} \]

Its Jacobian matrix is

\[ J_f = \begin{bmatrix} -u^2 + v^2 + 1 & 2uv \\ -2uv & -u^2 + v^2 - 1 \\ 2u & -2v \end{bmatrix} \]

Its metric then works out to be just a scalar function times the usual metric of the Euclidean plane:

\[ I = J_f^T J_f = (u^2 + v^2 + 1)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

This function is called the conformal scale factor.
Gauss Map
A vector is **normal** to a surface if it is orthogonal to all tangent vectors.

**Q:** Is there a **unique** normal at a given point?  

**A:** No! Can have different magnitudes/directions.

The **Gauss map** is a **continuous** map taking each point on the surface to a **unit** normal vector.

Can visualize Gauss map as a map from the surface to the unit sphere.

\[ \forall X, \langle N, df(X) \rangle = 0 \]
Orientability

Not every surface admits a Gauss map (globally):

orientable

nonorientable
Gauss Map—Example

Can obtain unit normal by taking the cross product of two tangents*:

\[ f := (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v)) \]

\[ df = \begin{pmatrix} -\sin(u) \sin(v), & \cos(u) \sin(v), & 0 \\ \cos(u) \cos(v), & \cos(v) \sin(u), & -\sin(v) \end{pmatrix} du + \begin{pmatrix} \cos(u) \sin^2(v) \\ -\sin(u) \sin^2(v) \\ -\cos(v) \sin(v) \end{pmatrix} dv \]

\[ df \left( \frac{\partial}{\partial u} \right) \times df \left( \frac{\partial}{\partial v} \right) = \begin{bmatrix} -\cos(u) \sin^2(v) \\ -\sin(u) \sin^2(v) \\ -\cos(v) \sin(v) \end{bmatrix} \]

To get unit normal, divide by length. In this case, can just notice we have a constant multiple of the sphere itself:

\[ \Rightarrow N = -f \]

*Must not be parallel!
Surjectivity of Gauss Map

• Given a unit vector $u$, can we always find some point on a surface that has this normal? ($N = u$)

• Yes! **Proof** (Hilbert):

Q: Is the Gauss map **injective**?
Vector Area

• Given a little patch of surface \( \Omega \), what’s the “average normal”?
• Can simply integrate normal over the patch, divide by area:

\[
\frac{1}{\text{area}(\Omega)} \int_{\Omega} N \, dA
\]

• Integrand \( N \, dA \) is called the vector area. (Vector-valued 2-form)
• Can be easily expressed via exterior calculus*:

\[
df \wedge df (X, Y) = df (X) \times df (Y) - df (Y) \times df (X) = 2df (X) \times df (Y) = 2NdA (X, Y)
\]

\[\implies \mathcal{A} = \frac{1}{2} df \wedge df\]
By expressing vector area this way, we make an interesting observation:

\[ 2 \int_{\Omega} N \, dA = \int_{\Omega} df \wedge df = \int_{\Omega} d(f \, df) = \int_{\partial \Omega} df = \int_{\partial \Omega} f(s) \times df(T(s)) \, ds \]

Hence, vector area is the same for any two patches with the same boundary.

Can define “normal” given only boundary (e.g., nonplanar polygon).

**Corollary:** integral of normal vanishes for any closed surface.
Curvature
The Weingarten map $dN$ is the differential of the Gauss map $N$.

At each point, tells us the change in the normal vector along any given direction $X$.

Since change in unit normal cannot have any component in the normal direction, $dN(X)$ is always tangent to the surface.

Can also think of it as a vector tangent to the unit sphere $S^2$.

Q: Why is $dN(Y)$ “flipped”? 
Weingarten Map — Example

• Recall that for the sphere, \( N = -f \). Hence, Weingarten map \( dN \) is just \(-df\):

\[
f := (\cos(u) \sin(v), \sin(u) \sin(v), \cos(v))
\]

\[
df = \begin{pmatrix}
-\sin(u) \sin(v), & \cos(u) \sin(v), & 0 \\
\cos(u) \cos(v), & \cos(v) \sin(u), & -\sin(v)
\end{pmatrix}
\]

\[
du + \begin{pmatrix}
\cos(u) \sin(v), & -\cos(u) \sin(v), & 0 \\
-\cos(u) \cos(v), & -\cos(v) \sin(u), & \sin(v)
\end{pmatrix}
dv
\]

\[
dN = \begin{pmatrix}
\sin(u) \sin(v), & -\cos(u) \sin(v), & 0 \\
-\cos(u) \cos(v), & -\cos(v) \sin(u), & \sin(v)
\end{pmatrix}
\]

Key idea: computing the Weingarten map is no different from computing the differential of a surface.
Normal Curvature

• For curves, curvature was the rate of change of the tangent; for immersed surfaces, we’ll instead consider how quickly the normal is changing.*

• In particular, normal curvature is rate at which normal is bending along a given tangent direction:

\[ \kappa_N(X) := \frac{\langle df(X), dN(X) \rangle}{|df(X)|^2} \]

• Equivalent to intersecting surface with normal-tangent plane and measuring the usual curvature of a plane curve

*For plane curves, what would happen if we instead considered change in \( N \)?
**Normal Curvature—Example**

Consider a parameterized cylinder:

\[ f(u, v) := (\cos(u), \sin(u), v) \]

\[ df = (-\sin(u), \cos(u), 0)\,du + (0, 0, 1)\,dv \]

\[ N = (-\sin(u), \cos(u), 0) \times (0, 0, 1) \]

\[ = (\cos(u), \sin(u), 0) \]

\[ dN = (-\sin(u), \cos(u), 0)\,du \]

\[ \kappa_N \left( \frac{\partial}{\partial u} \right) = \frac{\langle df \left( \frac{\partial}{\partial u} \right), dN \left( \frac{\partial}{\partial u} \right) \rangle}{|df \left( \frac{\partial}{\partial u} \right)|^2} = \frac{(-\sin(u), \cos(u), 0)\cdot(-\sin(u), \cos(u), 0)}{|(-\sin(u), \cos(u), 0)|^2} = 1 \]

\[ \kappa_N \left( \frac{\partial}{\partial v} \right) = \cdots = 0 \]

**Q:** Does this result make sense geometrically?
**Principal Curvature**

- Among all directions $X$, there are two **principal directions** $X_1, X_2$ where normal curvature has minimum/maximum value (respectively).

- Corresponding normal curvatures are the **principal curvatures**

- Two critical facts*:
  1. $g(X_1, X_2) = 0$
  2. $dN(X_i) = \kappa_i df(X_i)$

Where do these relationships come from?
Shape Operator

• The change in the normal $N$ is always tangent to the surface

• Must therefore be some linear map $S$ from tangent vectors to tangent vectors, called the shape operator, such that

$$df(SX) = dN(X)$$

• Principal directions are the eigenvectors of $S$

• Principal curvatures are eigenvalues of $S$

• Note: $S$ is not a symmetric matrix! Hence, eigenvectors are not orthogonal in $R^2$; only orthogonal with respect to induced metric $g$. 
Shape Operator—Example

Consider a nonstandard parameterization of the cylinder (sheared along z):
\[ f(u, v) := (\cos(u), \sin(u), u + v) \]
\[ df = (-\sin(u), \cos(u), 1)du + (0, 0, 1)dv \]
\[ N = (\cos(u), \sin(u), 0) \]
\[ dN = (-\sin(u), \cos(u), 0)du \]

\[ df \circ S = dN \]
\[ \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix} \]

\[ \Rightarrow S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \]
\[ X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

\[ df(X_1) = (0, 0, 1) \]
\[ \kappa_1 = 0 \]
\[ df(X_2) = (\sin(u), -\cos(u), 0) \]
\[ \kappa_2 = 1 \]

Key observation: principal directions orthogonal only in \( R^3 \).
Umbilic Points

• Points where principal curvatures are equal are called **umbilic points**

• Principal *directions* are not uniquely determined here

• What happens to the shape operator $S$?
  
  • May still have full rank!
  
  • Just have repeated eigenvalues, 2-dim. eigenspace

\[
S = \begin{bmatrix}
1/r & 0 \\
0 & 1/r
\end{bmatrix} \quad \kappa_1 = \kappa_2 = \frac{1}{r} \quad \forall X, \ SX = \frac{1}{r}X
\]

Could still of course choose (arbitrarily) an orthonormal pair $X_1, X_2...$
Principal Curvature Nets

- Walking along principal direction field yields \textit{principal curvature lines}.
- Collection of all such lines is called the \textit{principal curvature network}.
Separatrices and Spirals

- If we walk along a principal curvature line, where do we end up?
- Sometimes, a curvature line terminates at an umbilic point in both directions; these so-called separatrices (can) split network into regular patches.
- Other times, we make a closed loop. More often, however, behavior is not so nice!
Application—Quad Remeshing

- Recent approach to meshing: construct net *roughly* aligned with principal curvature—but with separatrices & loops, not spirals.

from Knöppel, Crane, Pinkall, Schröder, "Stripe Patterns on Surfaces"
Gaussian and Mean Curvature

Gaussian and mean curvature also fully describe local bending:

\[
\begin{align*}
\text{Gaussian} & \quad K := \kappa_1 \kappa_2 \\
\text{mean*} & \quad H := \frac{1}{2} (\kappa_1 + \kappa_2)
\end{align*}
\]

- \( K > 0 \) \quad \text{“developable”} \quad K = 0
- \( H \neq 0 \) \quad H \neq 0 \quad \text{“minimal”} \quad H = 0

*Warning: another common convention is to omit the factor of 1/2
Total Mean Curvature?

**Theorem** (Minkowski): for a regular closed embedded surface,

\[ \int_M H \, dA \geq \sqrt{4\pi A} \]

**Q:** When do we get equality?

**A:** For a sphere.
Second Fundamental Form

• Second fundamental form is closely related to principal curvature

• Can also be viewed as change in first fundamental form under motion in normal direction

• Why “fundamental?” First & second fundamental forms play role in important theorem...

\[ \mathbf{II}(X,Y) := \langle dN(X), df(Y) \rangle \]

\[ \kappa_N(X) := \frac{df(X), dN(X)}{|df(X)|^2} = \frac{\mathbf{II}(X,X)}{I(X,X)} \]
Fundamental Theorem of Surfaces

- **Fact.** Two surfaces in $\mathbb{R}^3$ are congruent if and only if they have the same first and second fundamental forms.

- ...However, not every pair of bilinear forms $I, II$ on a domain $U$ describes a valid surface—must satisfy the **Gauss Codazzi** equations.

- Analogous to fundamental theorem of plane curves: determined up to rigid motion by curvature.

- ...However, for *closed* curves not every curvature function is valid (e.g., must integrate to $2k\pi$).