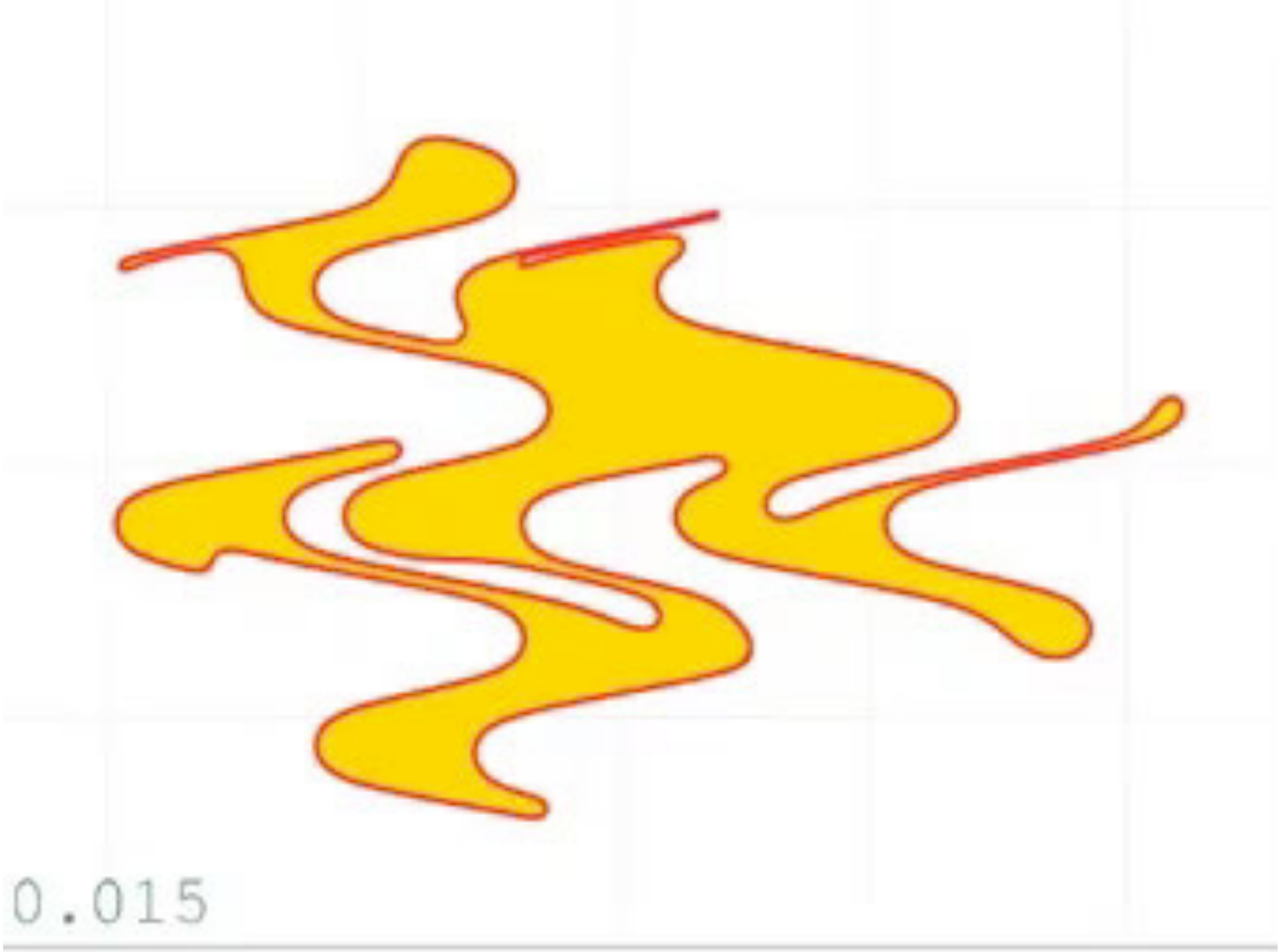
Length Shortening Flow

- The objective for length shortening flow is simply the total length of the curve; the flow is then the (L^2) gradient flow.
- For closed curves, several interesting features (Gage-Grayson-Hamilton):
    - Center of mass is preserved
    - Curves flow to "round points"
    - Embedded curves remain embedded

$$\mathrm{length}(\gamma) := \int_0^L |\tfrac{d}{ds}\gamma|\, ds$$

$$\tfrac{d}{dt}\gamma = -\nabla_\gamma \mathrm{length}(\gamma)$$

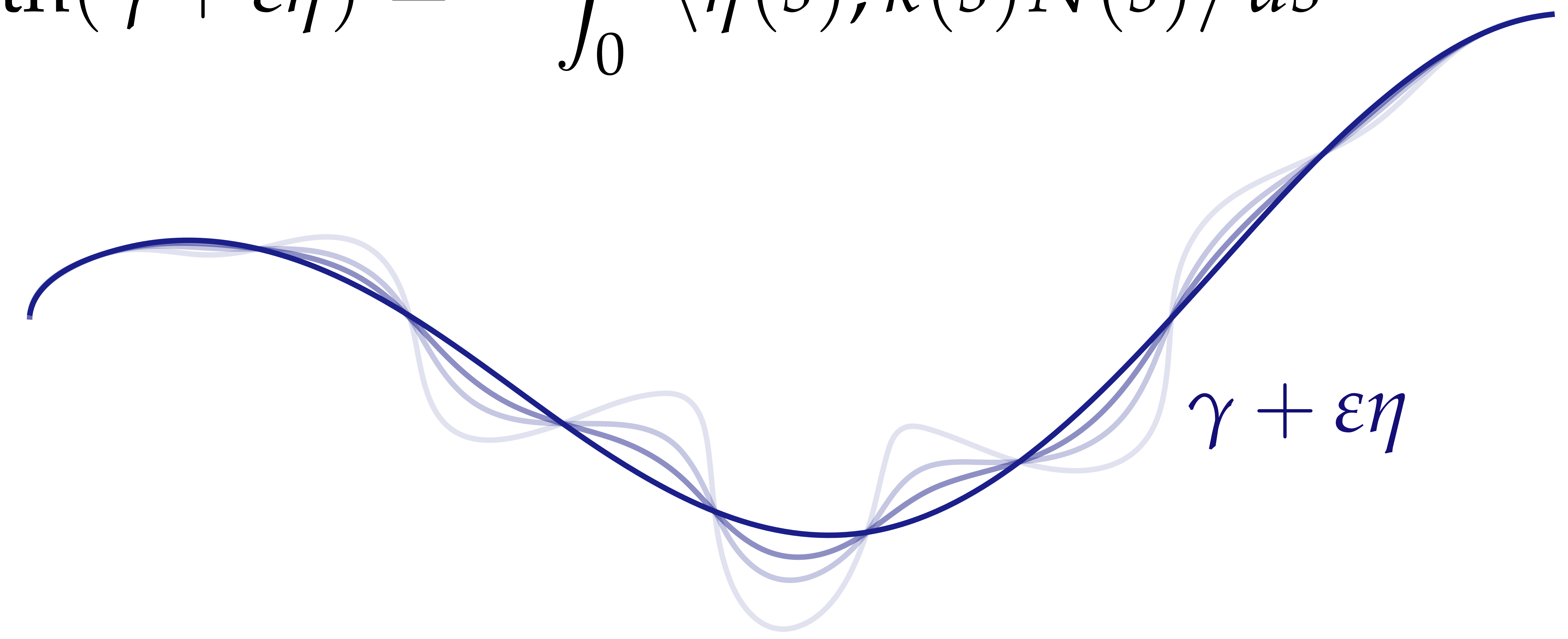


credit: Sigurd Angenent

Length Shortening Flow

Let length(γ) denote the total length of a regular plane curve $\gamma : [0, L] \to \mathbb{R}^2$, and consider a variation $\eta : [0, L] \to \mathbb{R}^2$ vanishing at endpoints. One can then show that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{length}(\gamma + \varepsilon\eta) = -\int_0^L \langle \eta(s), \kappa(s) N(s) \rangle \, ds$$



Key idea: quickest way to reduce length is to move in the direction κN.

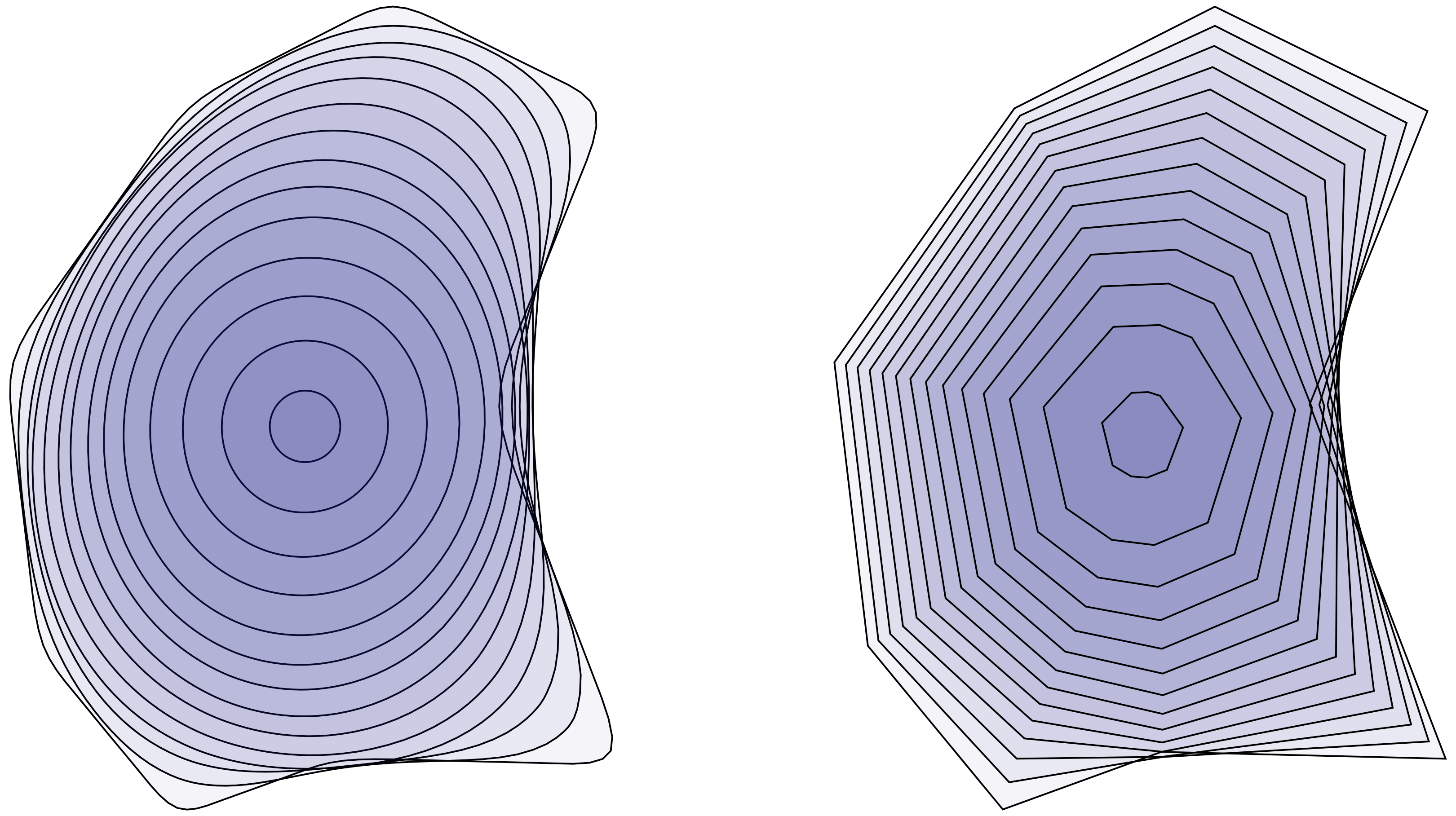
Length Shortening Flow—Forward Euler

- At each moment in time, move curve in normal direction with speed proportional to curvature
- "Smooths out" curve (*e.g.*, noise), eventually becoming circular
- Discretize by replacing time derivative with difference in time; smooth curvature with one (of many) curvatures
- Repeatedly add a little bit of κN (*"forward Euler method"*)

$$\frac{d}{dt}\gamma(s,t) = -\kappa(s,t)N(s,t)$$

$$\frac{\gamma_i^{t+1} - \gamma_i^t}{\tau} = -\kappa_i^t N_i^t$$

$$\Rightarrow \gamma_i^{t+1} = \gamma_i^t - \tau\kappa_i^t N_i^t$$



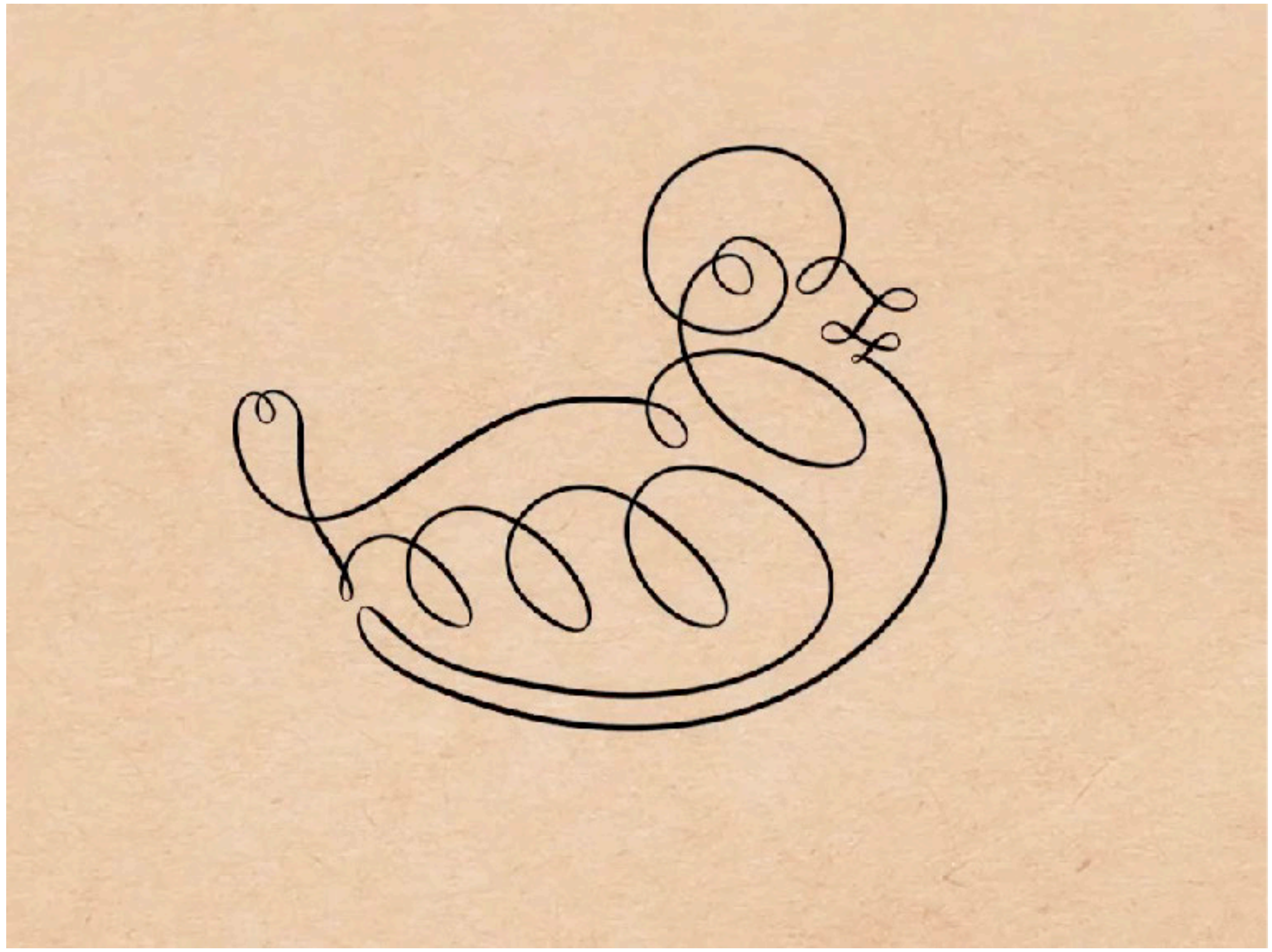
smooth **discrete**

Elastic Flow

- Basic idea: rather than shrinking length, try to reduce *bending* (curvature)
- Objective is integral of squared curvature; elastic flow is then gradient flow on this objective
- Minimizers are called *elastic curves*
- More interesting w/ constraints (e.g., endpoint positions & a tangents)

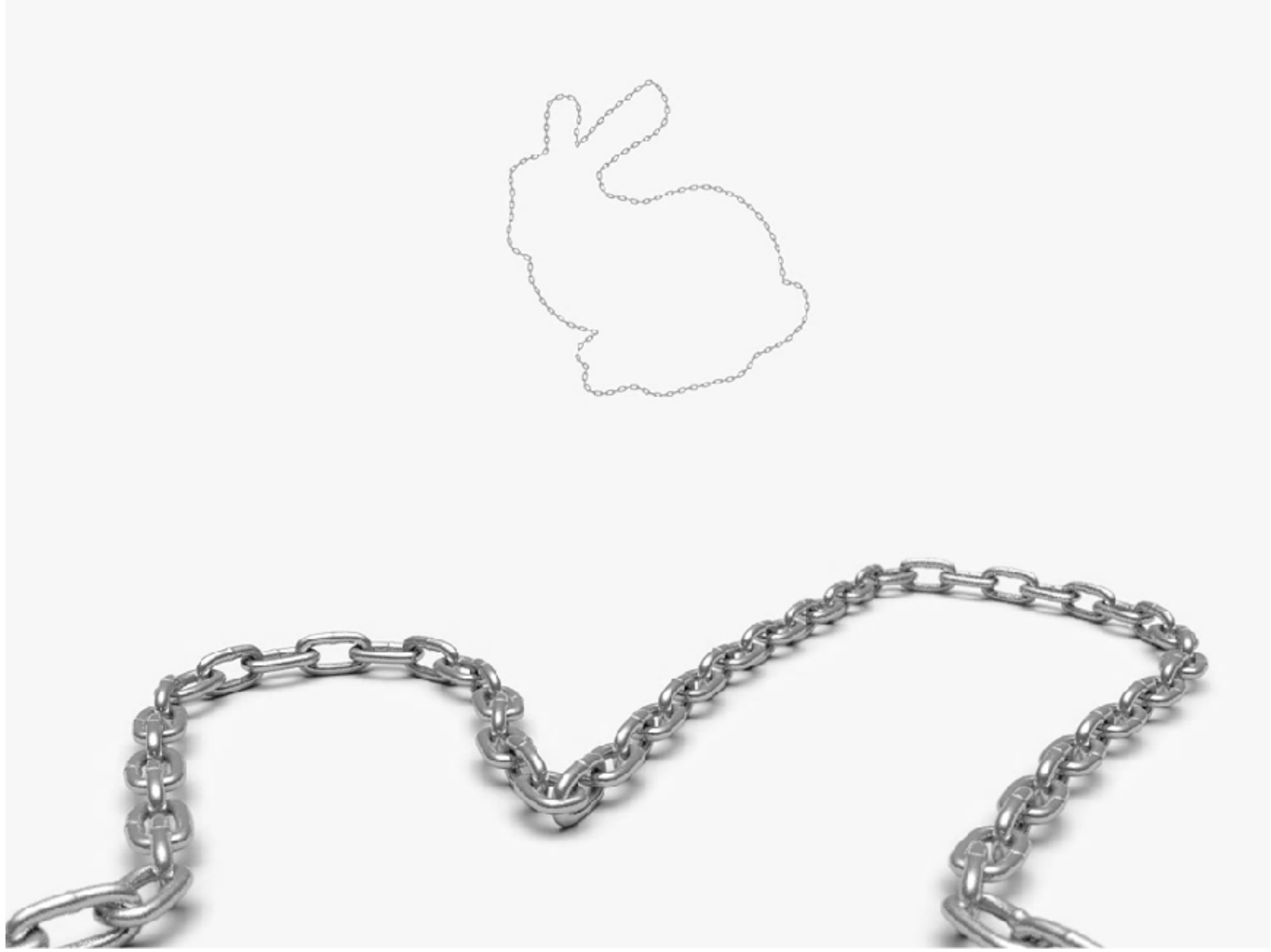
$$E(\gamma) := \int_0^L \kappa(s)^2 \, ds$$

$$\tfrac{d}{dt}\gamma = -\nabla_\gamma E(\gamma)$$



Isometric Elastic Flow

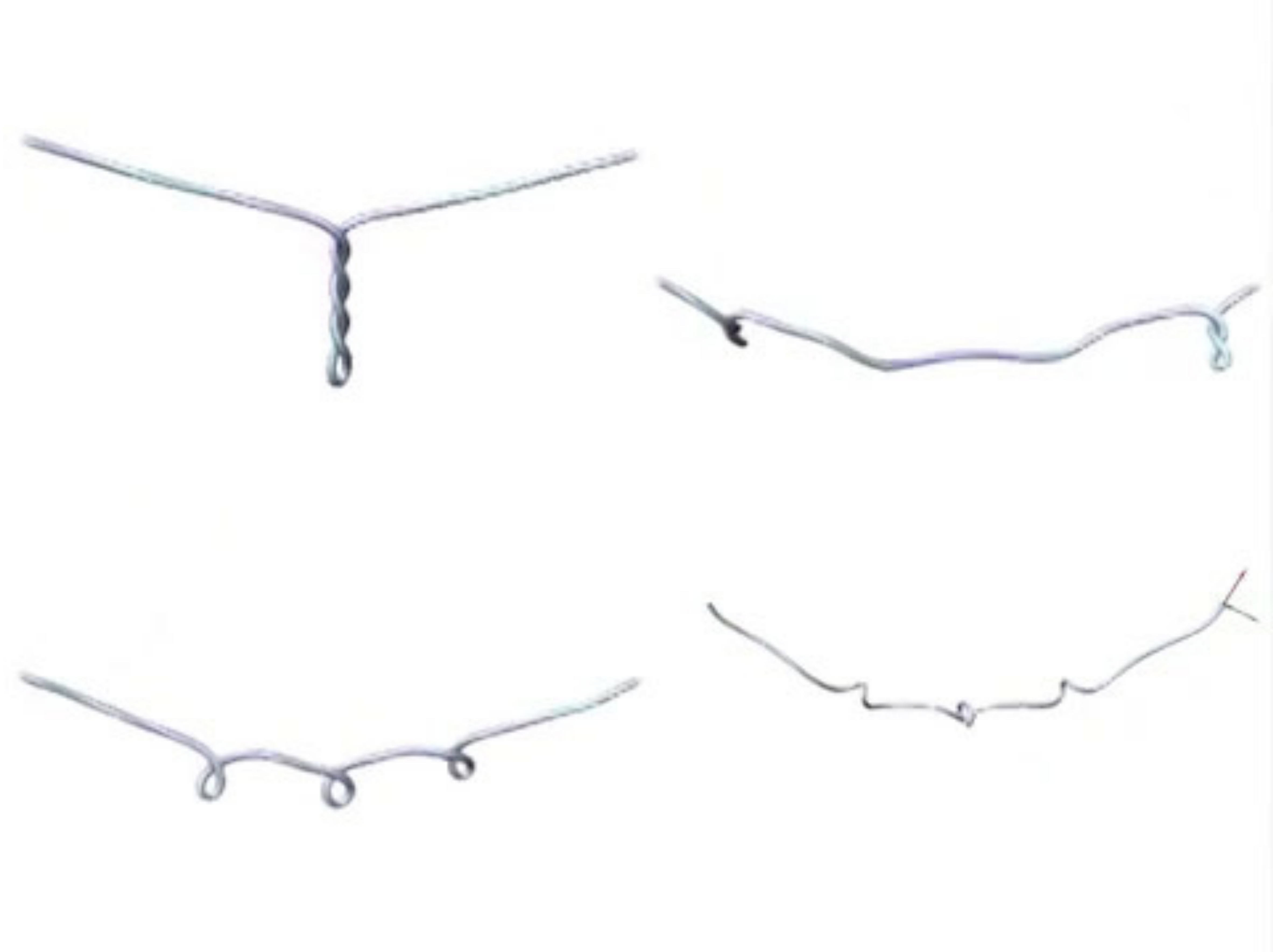
- Different way to smooth out a curve is to directly "shrink" curvature
- Discrete case: "scale down" turning angles, then use the fundamental theorem of discrete plane curves to reconstruct
- Extremely stable numerically; exactly preserves edge lengths
- Challenge: how do we make sure closed curves remain closed?



From Crane et al, *"Robust Fairing via Conformal Curvature Flow"*

Elastic Rods

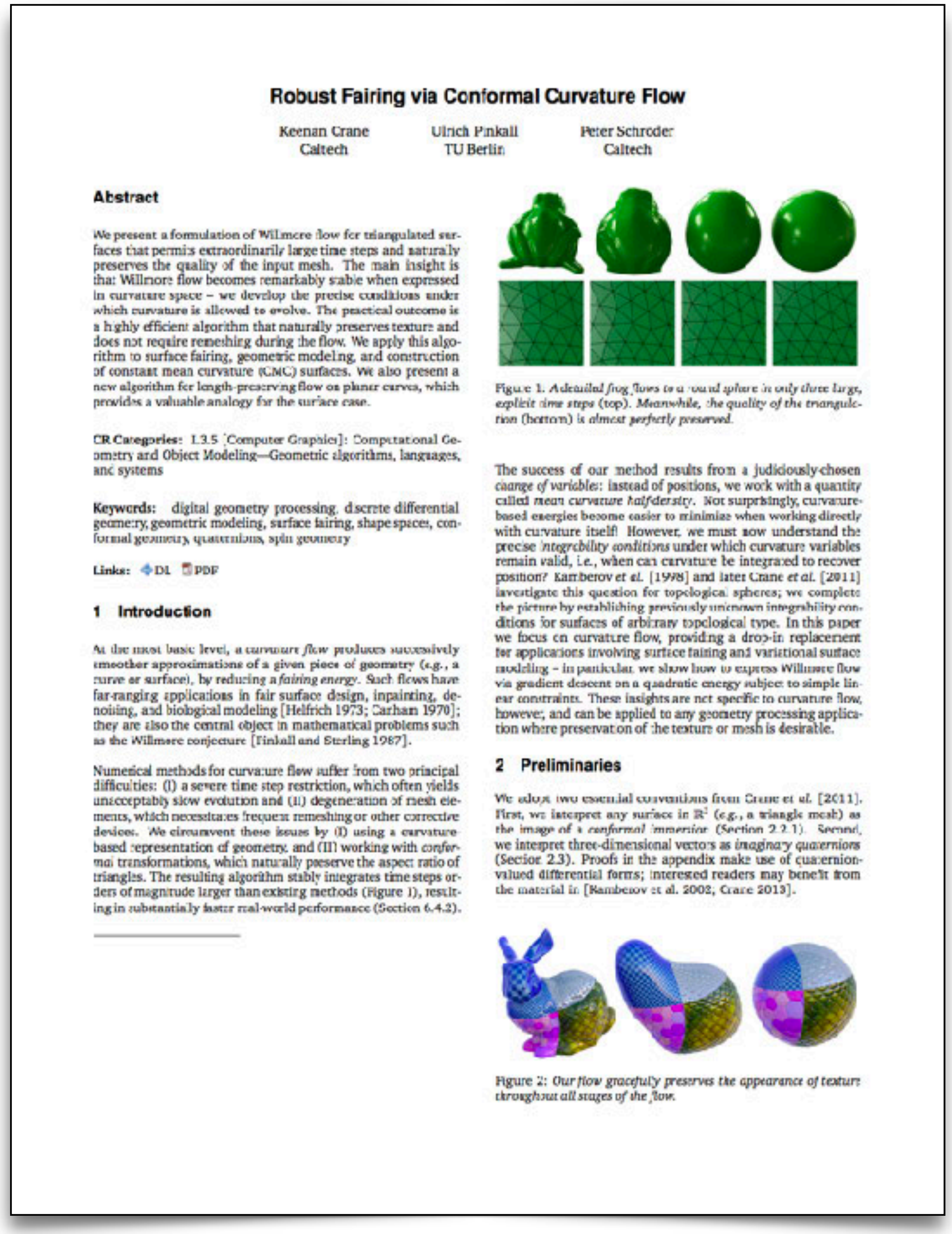
- For space curve, can also try to minimize both *curvature* **and** *torsion*
- Both in some sense measure "non-straightness" of curve
- Provides rich model of *elastic rods*
- Lots of interesting applications (simulating hair, laying cable, …)



From Bergou et al, *"Discrete Elastic Rods"*

Reading Assignment

- Readings from papers on curve algorithms (will be posted online)

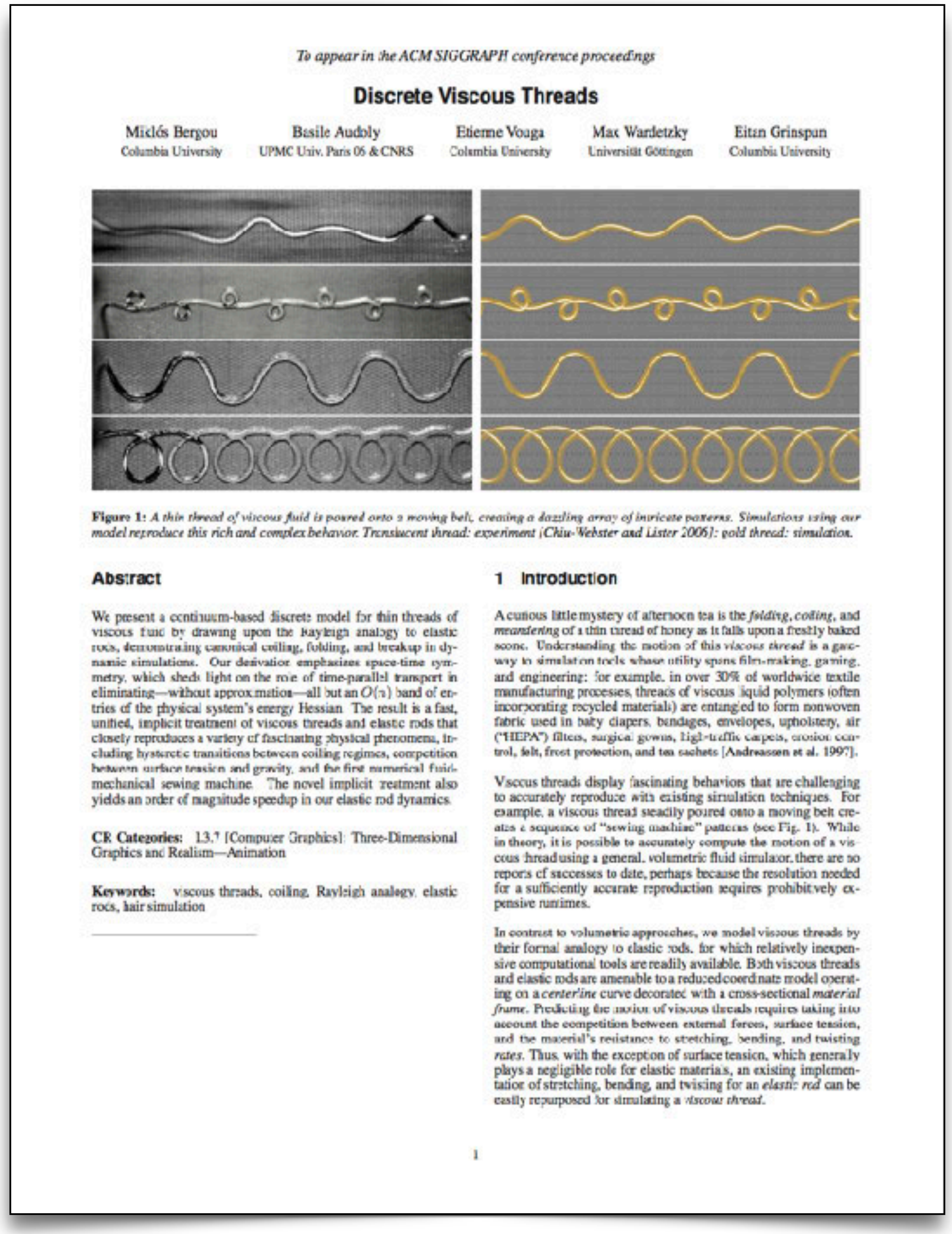


Robust Fairing via Conformal Curvature Flow

Keenan Crane, Caltech
Ulrich Pinkall, TU Berlin
Peter Schröder, Caltech

Discrete Elastic Rods

Miklós Bergou, Columbia University
Max Wardetzky, Freie Universität Berlin
Stephen Robinson, Columbia University
Basile Audoly, CNRS / UPMC Univ Paris 06
Eitan Grinspun, Columbia University



To appear in the ACM SIGGRAPH conference proceedings

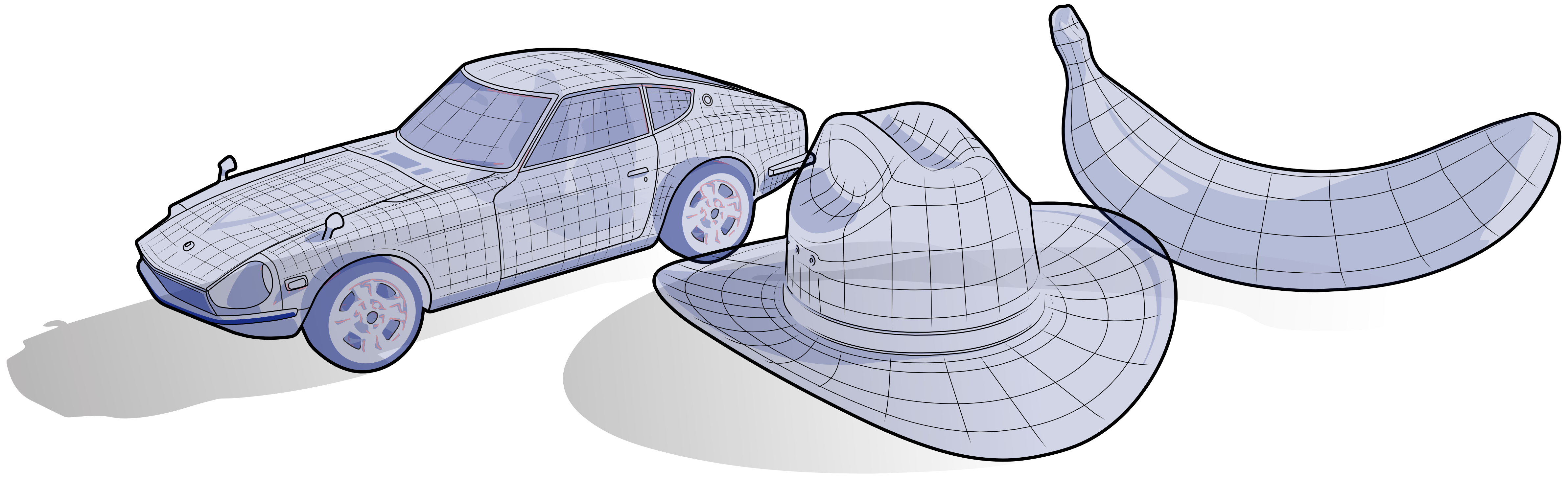
Discrete Viscous Threads

Miklós Bergou, Columbia University
Basile Audoly, UPMC Univ. Paris 06 & CNRS
Etienne Vouga, Columbia University
Max Wardetzky, Universität Göttingen
Eitan Grinspun, Columbia University

From Curves to Surfaces

- **Previously:** saw how to talk about 1D curves (both smooth and discrete)
- **Today:** will study 2D curved surfaces (both smooth and discrete)
  - Some concepts remain the same (*e.g.*, differential); others need to be generalized (*e.g.*, curvature)
  - Still use exterior calculus as our *lingua franca*

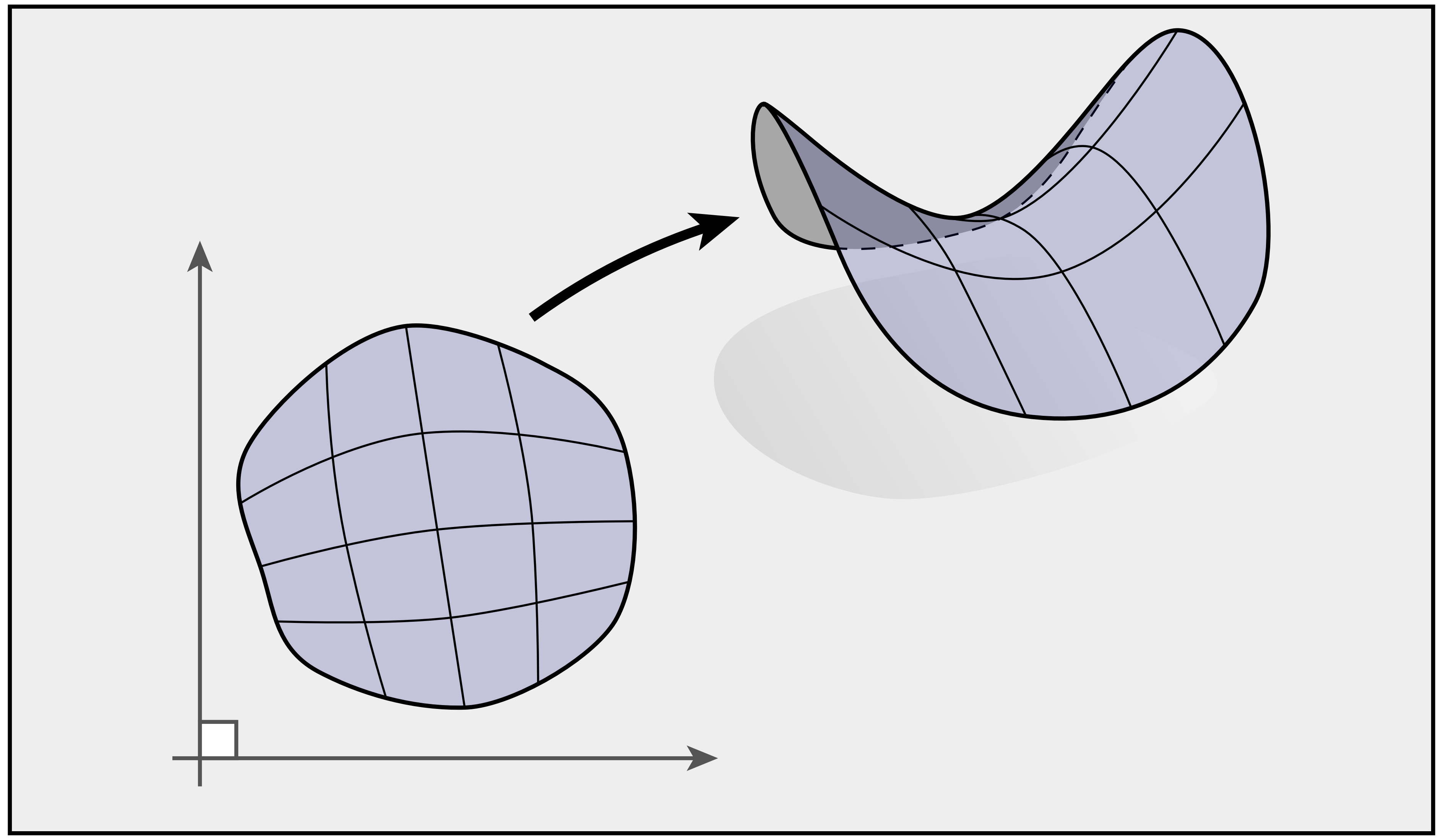
(Curves)

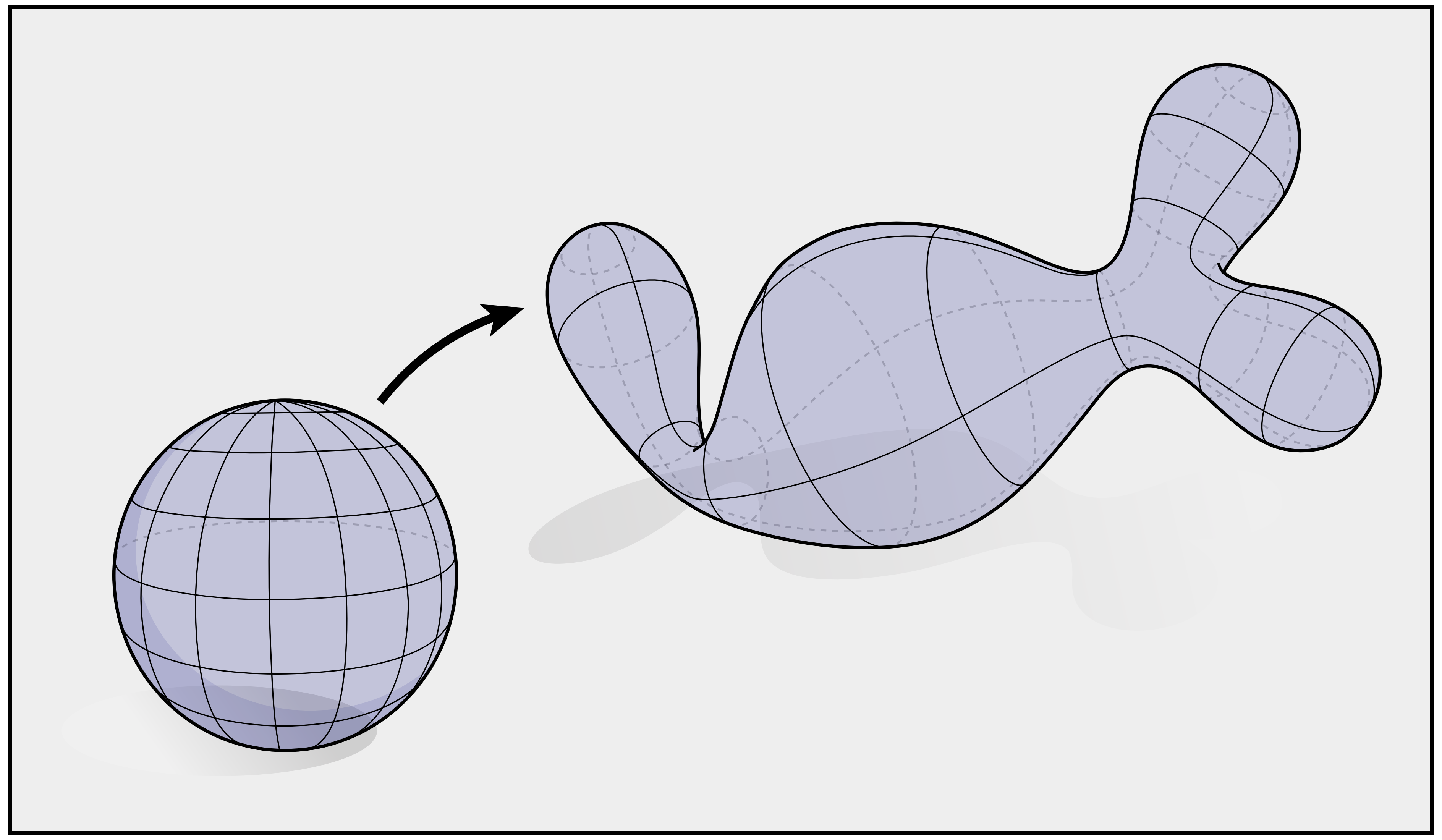


(Surfaces)

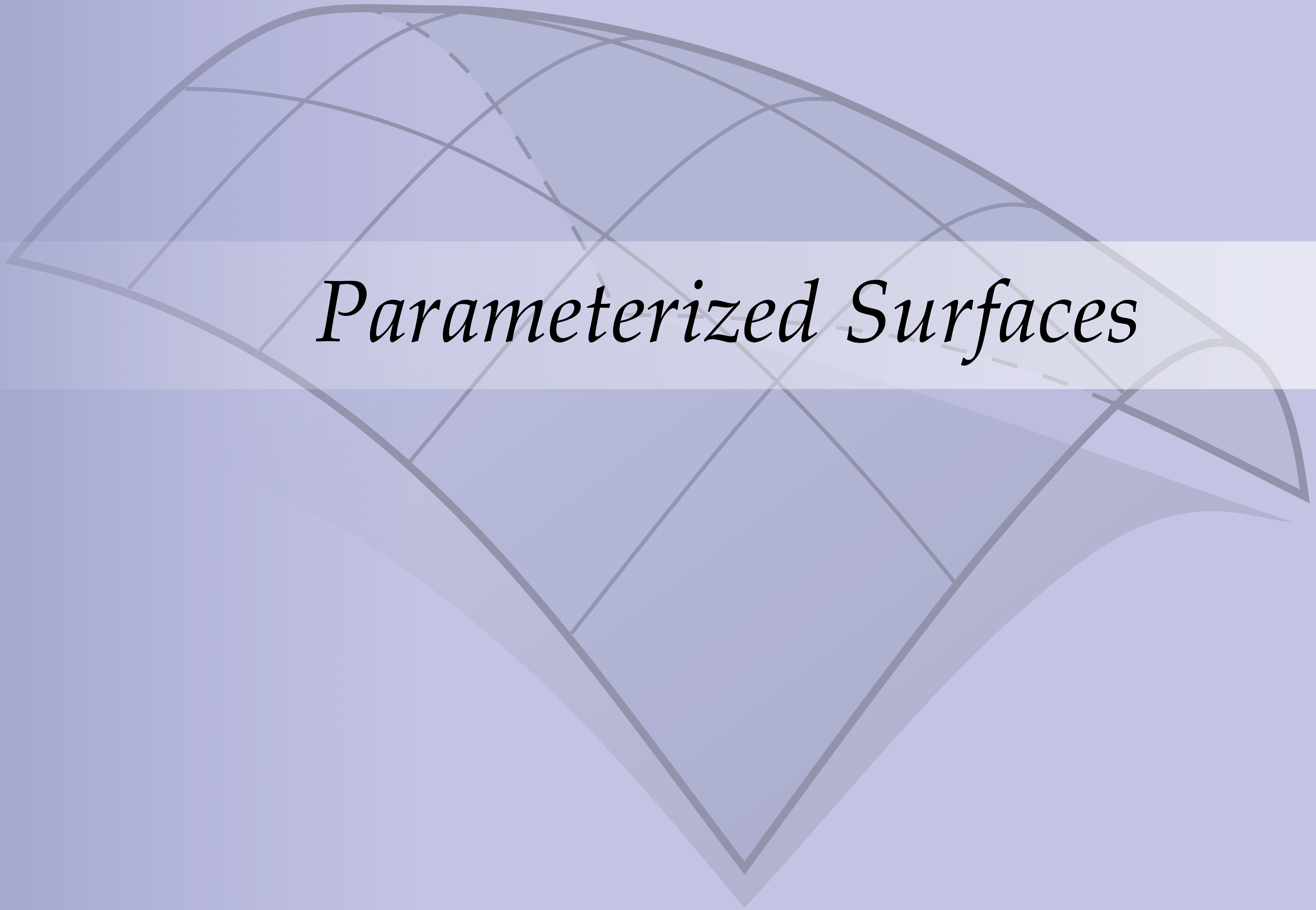
Surfaces—Local vs. Global View

- So far, we've only studied exterior calculus in R^n
- Will therefore be easiest to think of surfaces expressed in terms of patches of the plane (**local picture**)
- Later, when we study topology & smooth manifolds, we'll be able to more easily think about "whole surfaces" all at once (**global picture**)
- Global picture is *much* better model for **discrete** surfaces (meshes)…





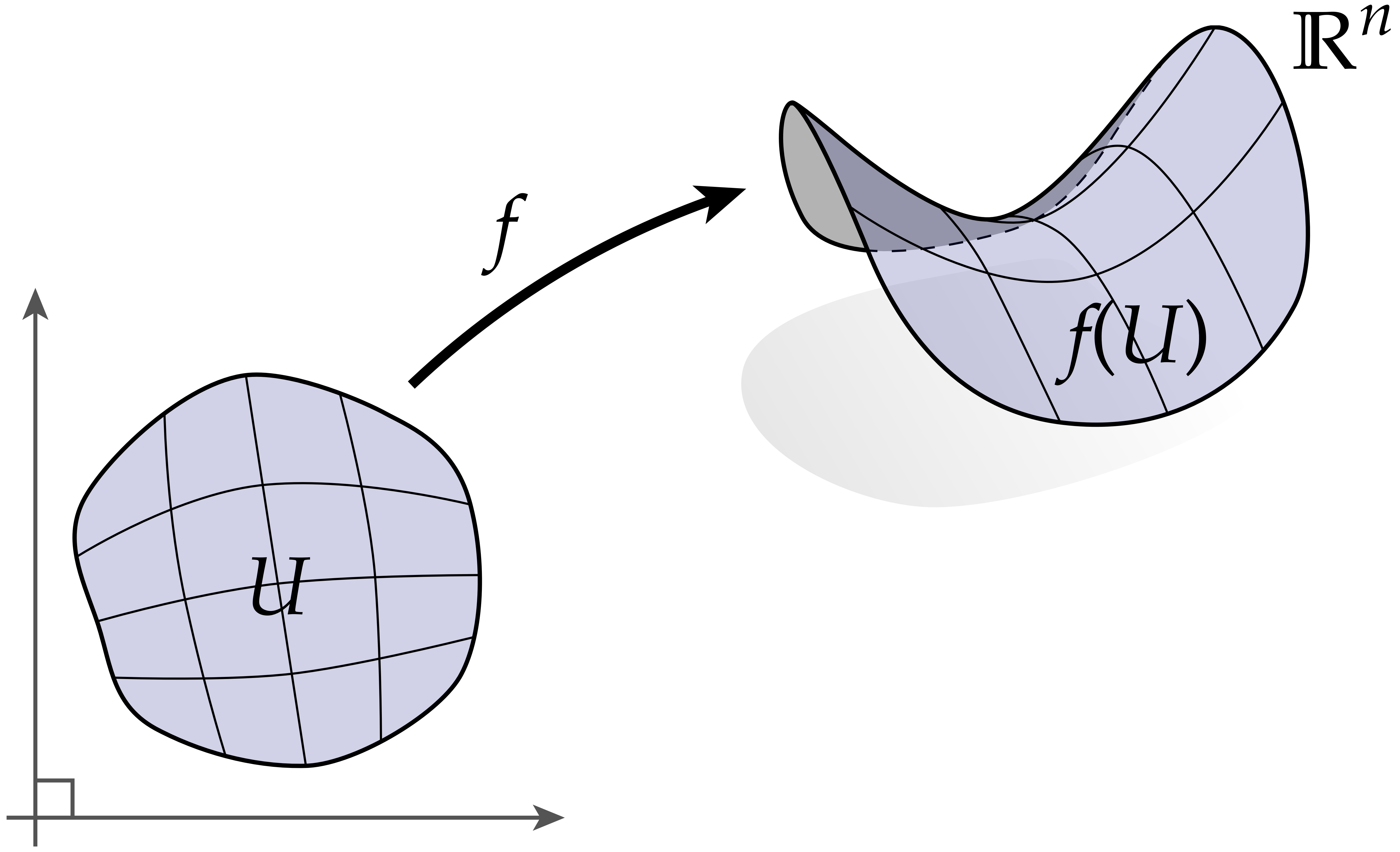
Parameterized Surfaces



Parameterized Surface

A **parameterized surface** is a map from a two-dimensional region $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$:

$$f : U \to \mathbb{R}^n$$



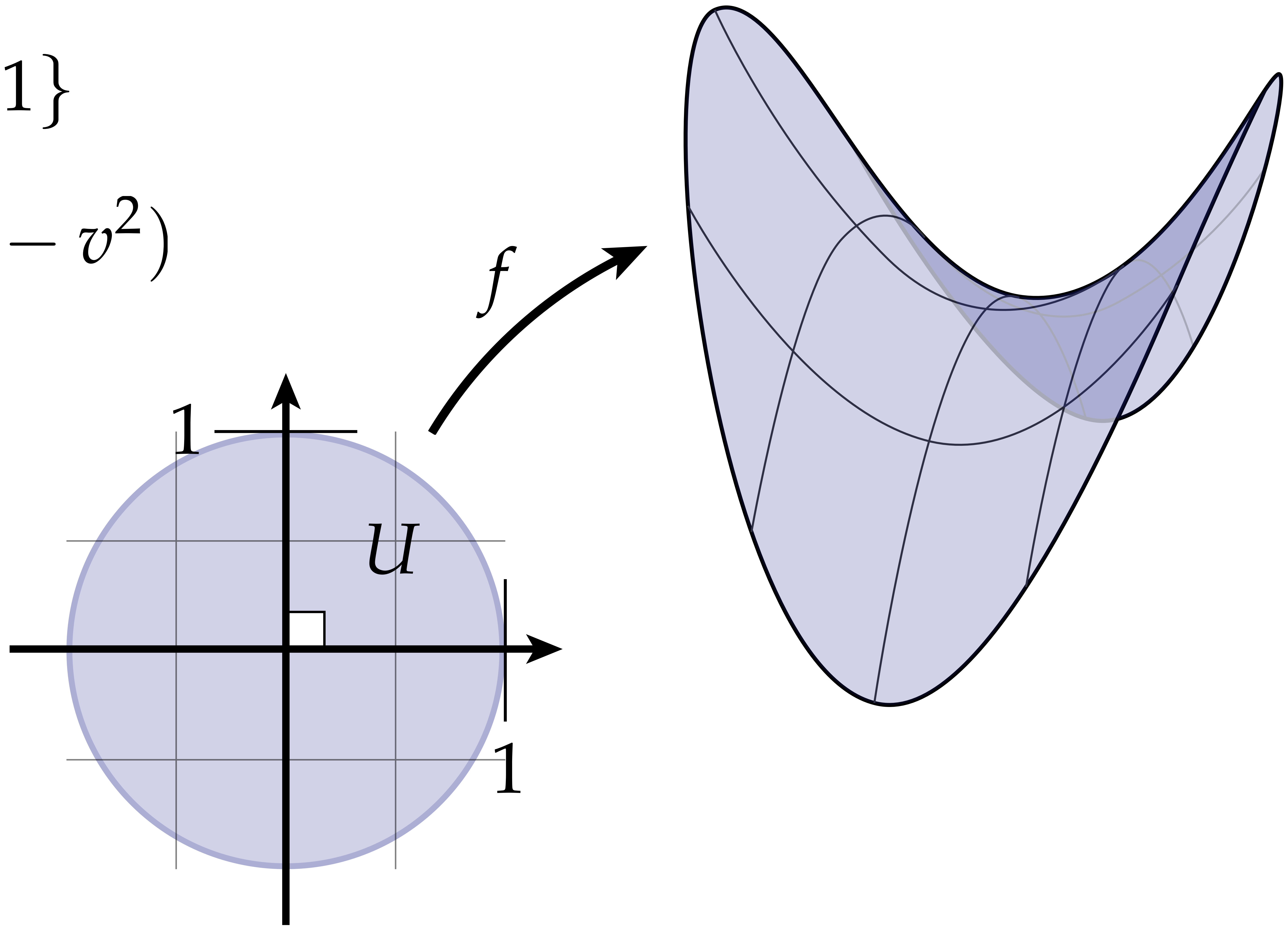
The set of points $f(U)$ is called the **image** of the parameterization.

Parameterized Surface—Example

- As an example, we can express a *saddle* as a parameterized surface:

$$U := \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$f : U \to \mathbb{R}^3;\ (u,v) \mapsto (u, v, u^2 - v^2)$$

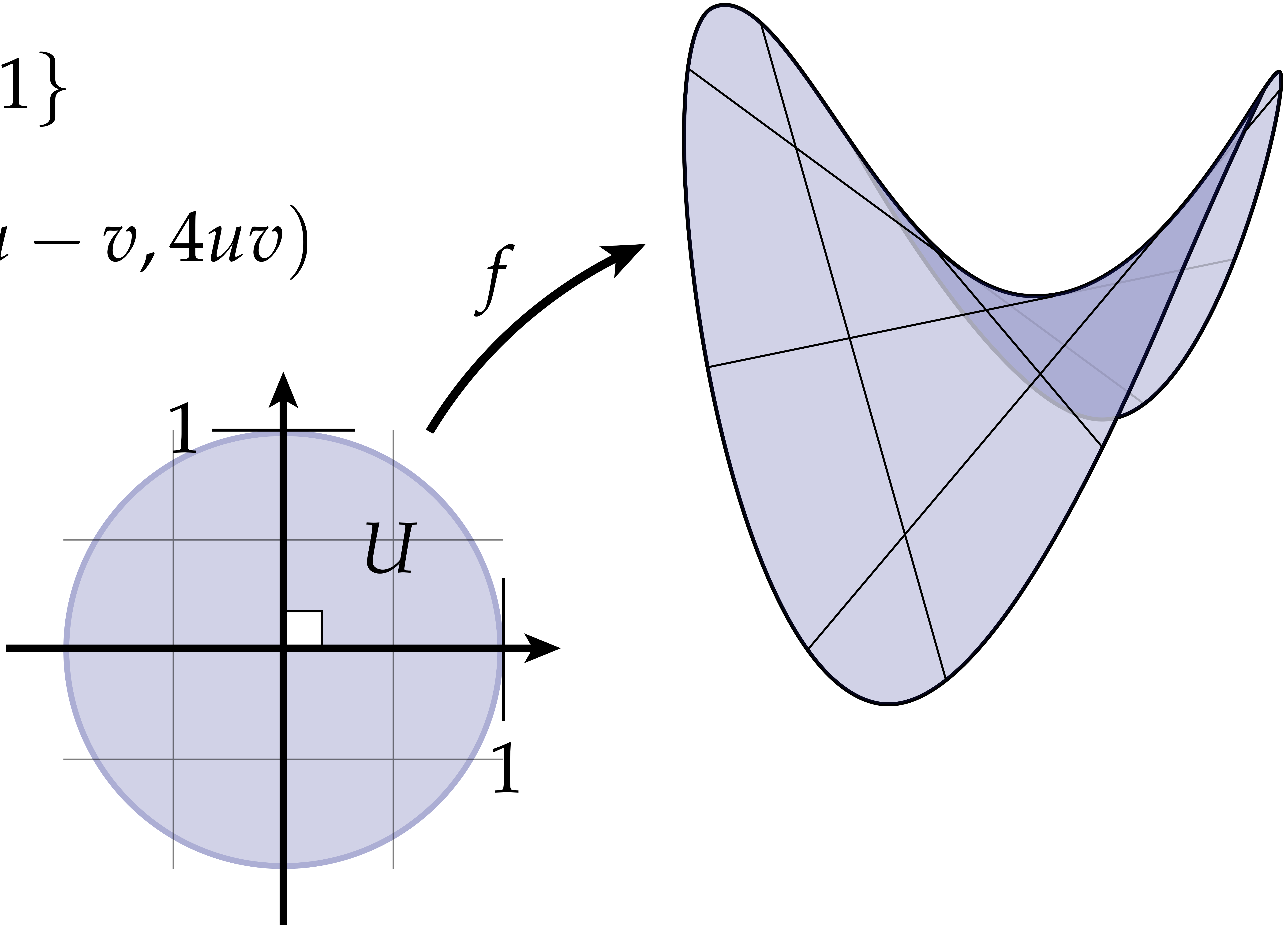


Reparameterization

- Many different parameterized surfaces can have the same image:

$$U := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$$

$$f : U \to \mathbb{R}^3;\ (u, v) \mapsto (u + v, u - v, 4uv)$$

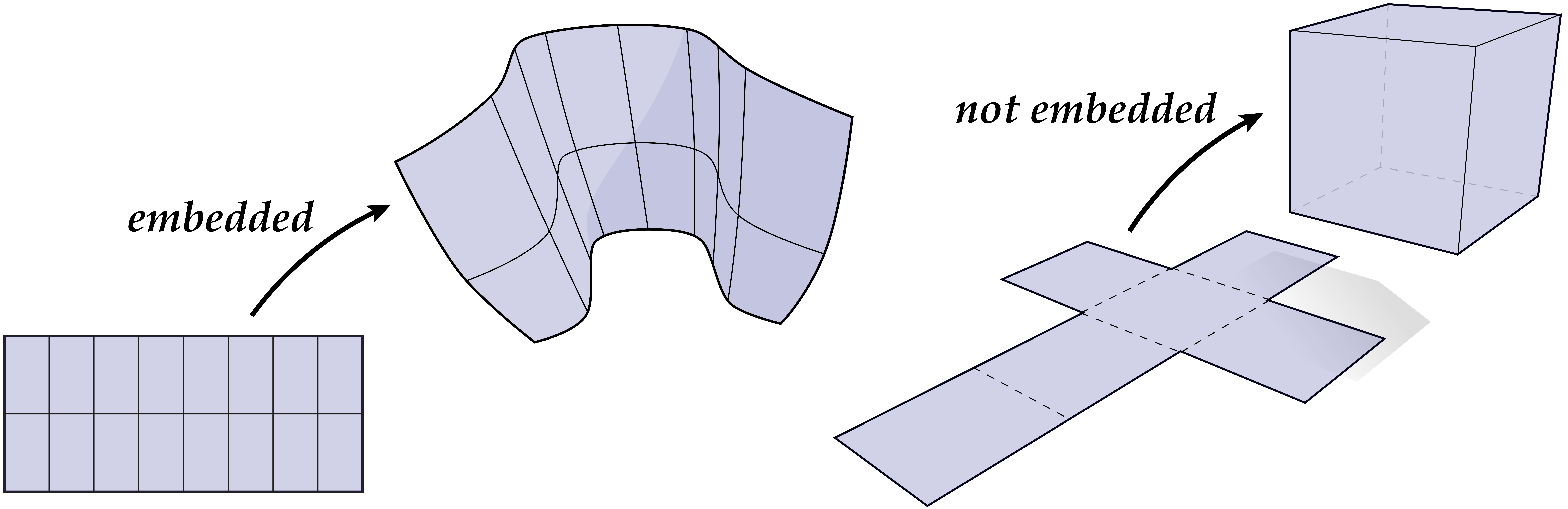


This *"reparameterization symmetry"* can be a major challenge in applications—*e.g.*, trying to decide if two parameterized surfaces (or meshes) describe the same shape.

Analogy: graph isomorphism

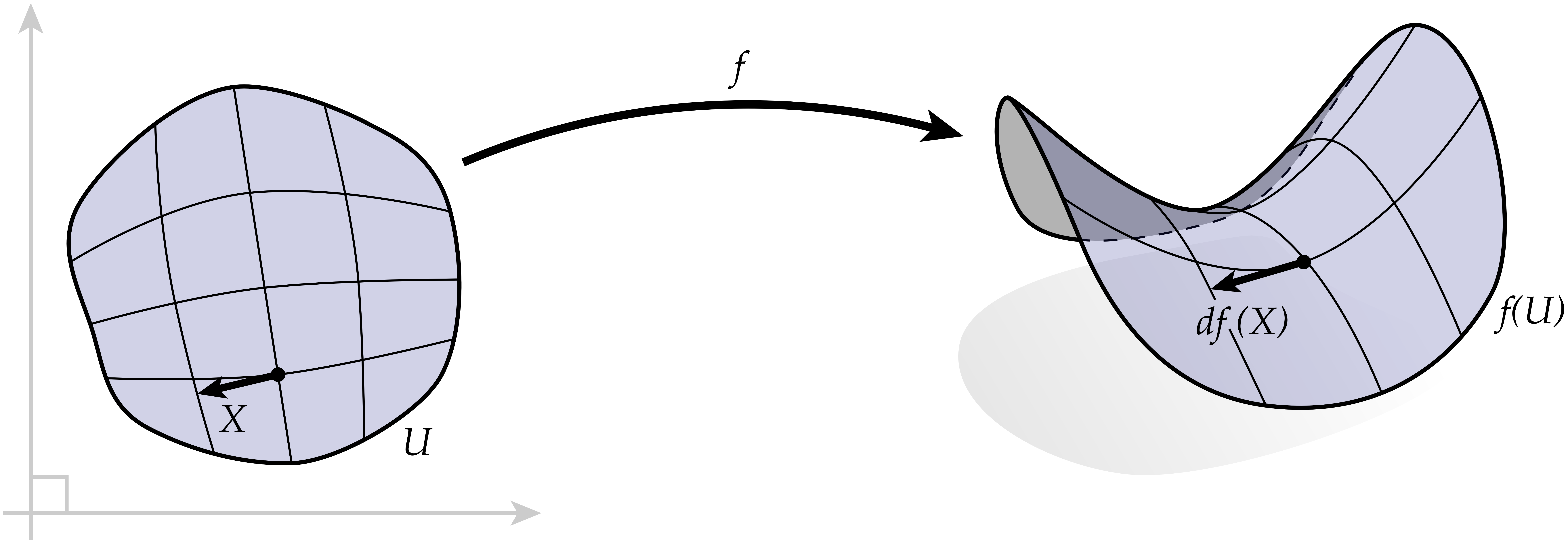
Embedded Surface

- Roughly speaking, an **embedded** surface does not self-intersect
- More precisely, a parameterized surface is an embedding if it is a continuous injective map, and has a continuous inverse on its image



Differential of a Surface

Intuitively, the *differential* of a parameterized surface tells us how tangent vectors on the domain get mapped to vectors in space:



We say that *df "pushes forward"* vectors X into R^n, yielding vectors $df(X)$

Differential in Coordinates

In coordinates, the differential is simply the exterior derivative:

$$f : U \to \mathbb{R}^3;\ (u, v) \mapsto (u, v, u^2 - v^2)$$

$$df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv =$$

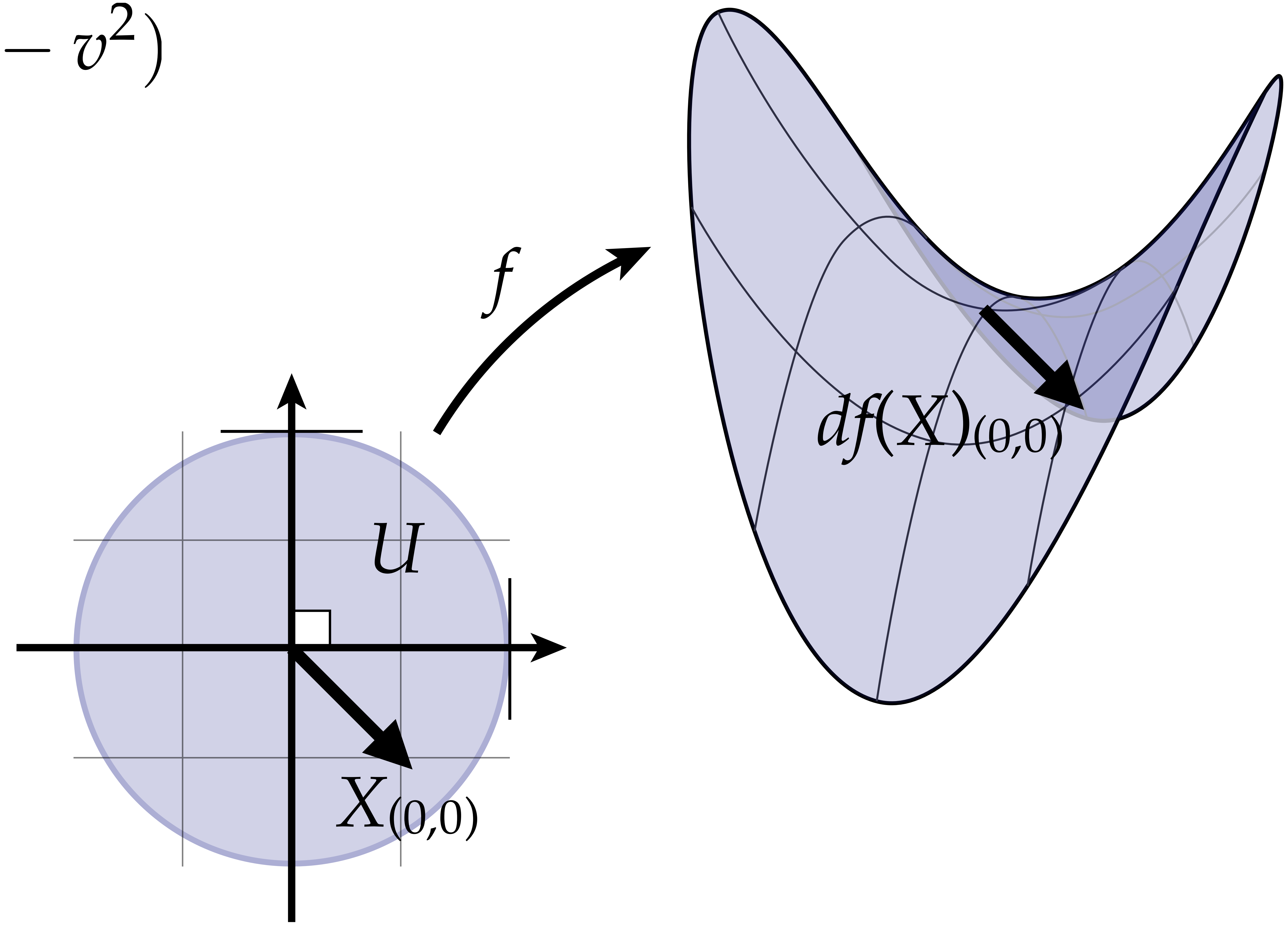
$$(1, 0, 2u)du + (0, 1, -2v)dv$$

<ins>Pushforward of a vector field:</ins>

$$X := \tfrac{3}{4}\left(\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial y}\right)$$

$$df(X) = \tfrac{3}{4}(1, -1, 2(u + v))$$

E.g., at u=v=0: $(\tfrac{3}{4}, -\tfrac{3}{4}, 0)$



Differential—Matrix Representation (Jacobian)

Definition. Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$, and let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. Then the *Jacobian* of f is the matrix

$$J_f := \begin{bmatrix} \partial f^1/\partial x^1 & \cdots & \partial f^1/\partial x^n \\ \vdots & \ddots & \vdots \\ \partial f^m/\partial x^1 & \cdots & \partial f^m/\partial x^n \end{bmatrix},$$

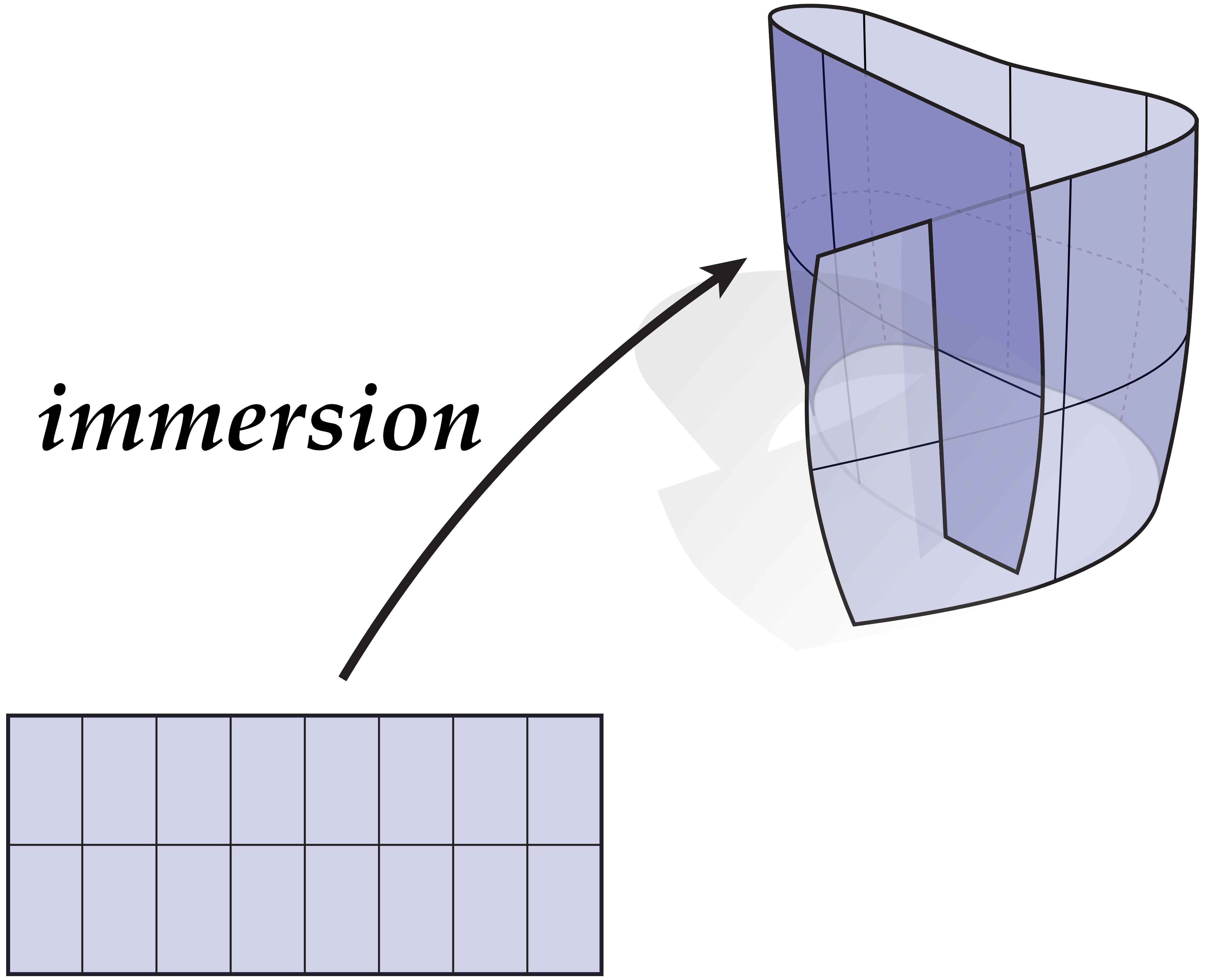
where $f^1, \ldots, f^m$ are the components of f w.r.t. some coordinate system on $\mathbb{R}^m$. This matrix represents the differential in the sense that $df(X) = J_f X$.

(In solid mechanics, also known as the *deformation gradient*.)

Note: does not generalize to infinite dimensions! (E.g., maps between functions.)

Immersed Surface

- A parameterized surface f is an *immersion* if its differential is nondegenerate, *i.e.*, if $df(X) = 0$ if and only if $X = 0$.



Intuition: no region of the surface gets “pinched”

Immersion—Example

Consider the standard parameterization of the sphere:

$$f(u,v) := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

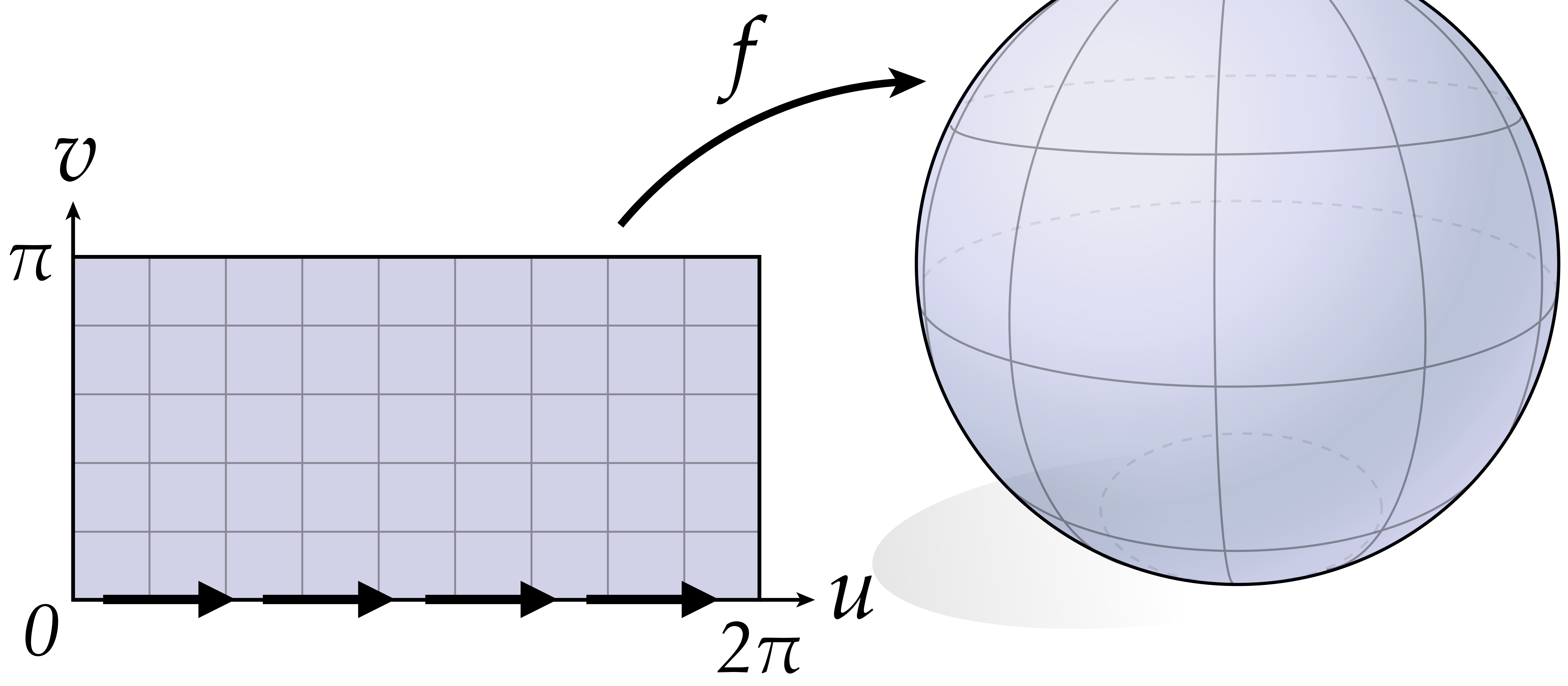
$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = \begin{array}{lccccl} (& -\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &) & du \, + \\ (& \cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &) & dv \end{array}$$

Q: Is f an immersion?

A: No: when $v = 0$ we get

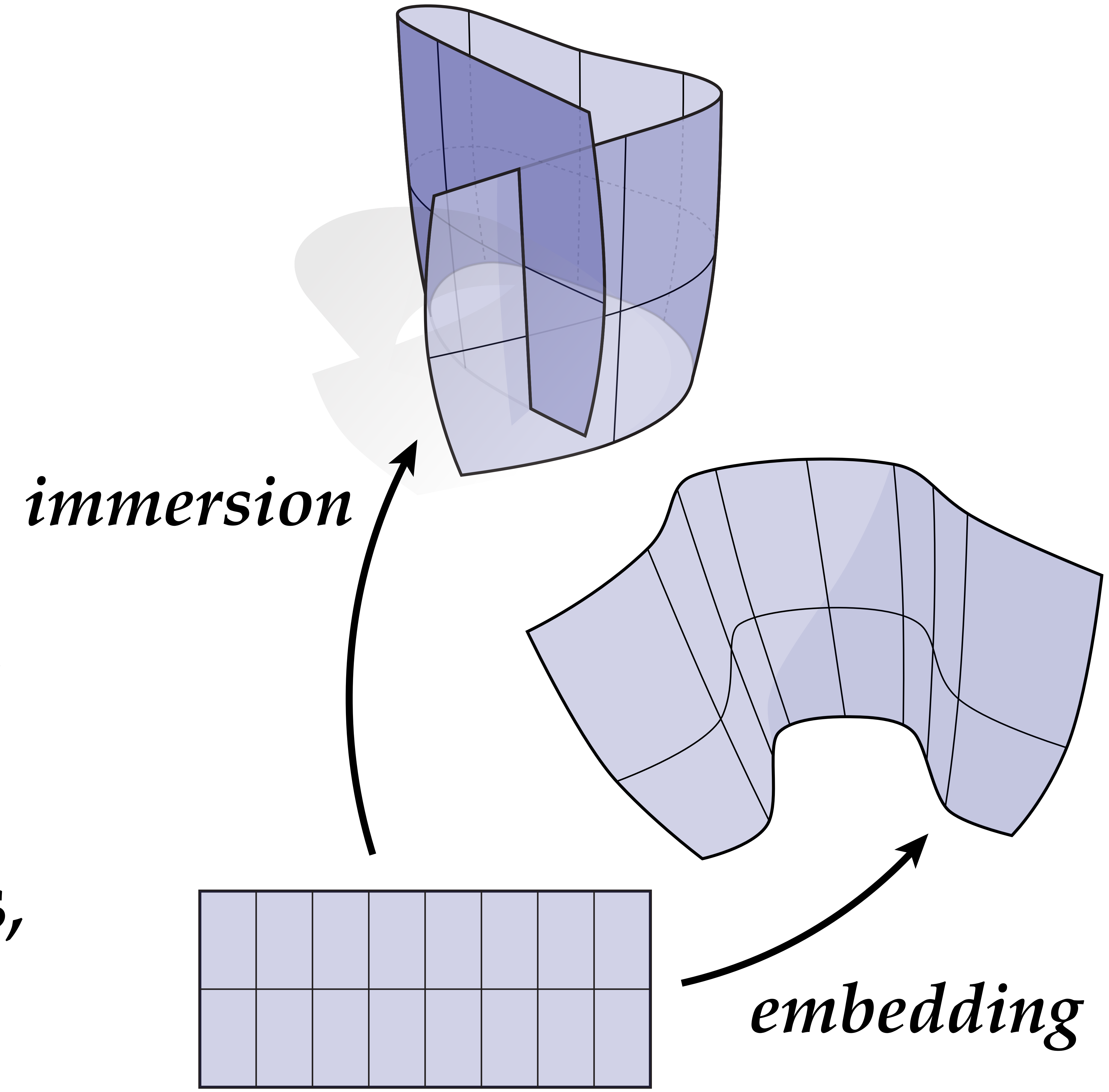
$$\begin{array}{lccccl} (& 0, & 0, & 0 &) & du \, + \\ (& \cos(u), & \sin(u), & -\sin(v) &) & dv \end{array}$$

Nonzero tangents mapped to zero!



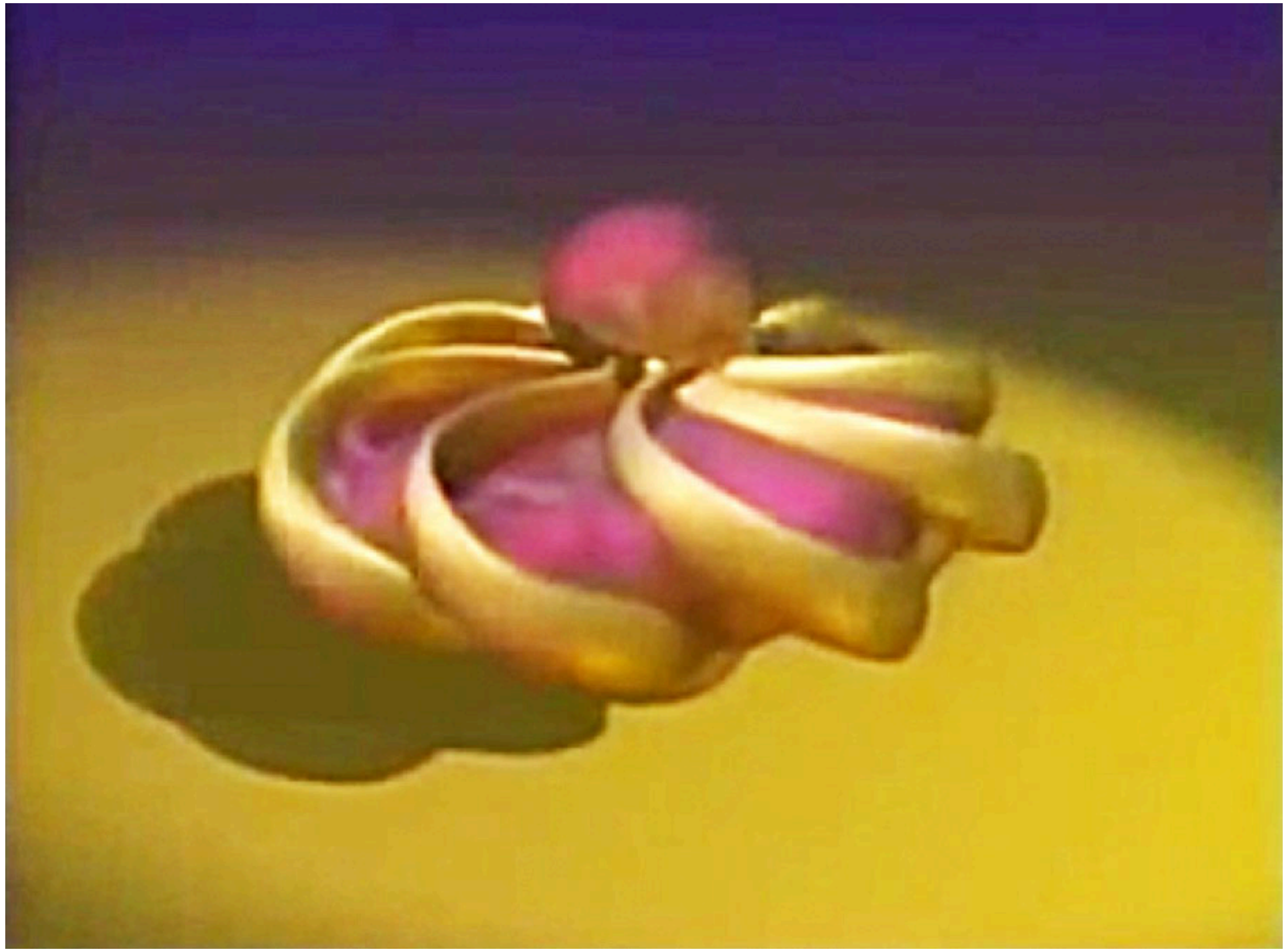
Immersion vs. Embedding

- In practice, ensuring that a surface is globally embedded can be challenging
- Immersions are typically "nice enough" to define local quantities like tangents, normals, metric, etc.
- Immersions are also a natural model for the way we typically think about meshes: most quantities (angles, lengths, etc.) are perfectly well-defined, even if there happen to be self-intersections



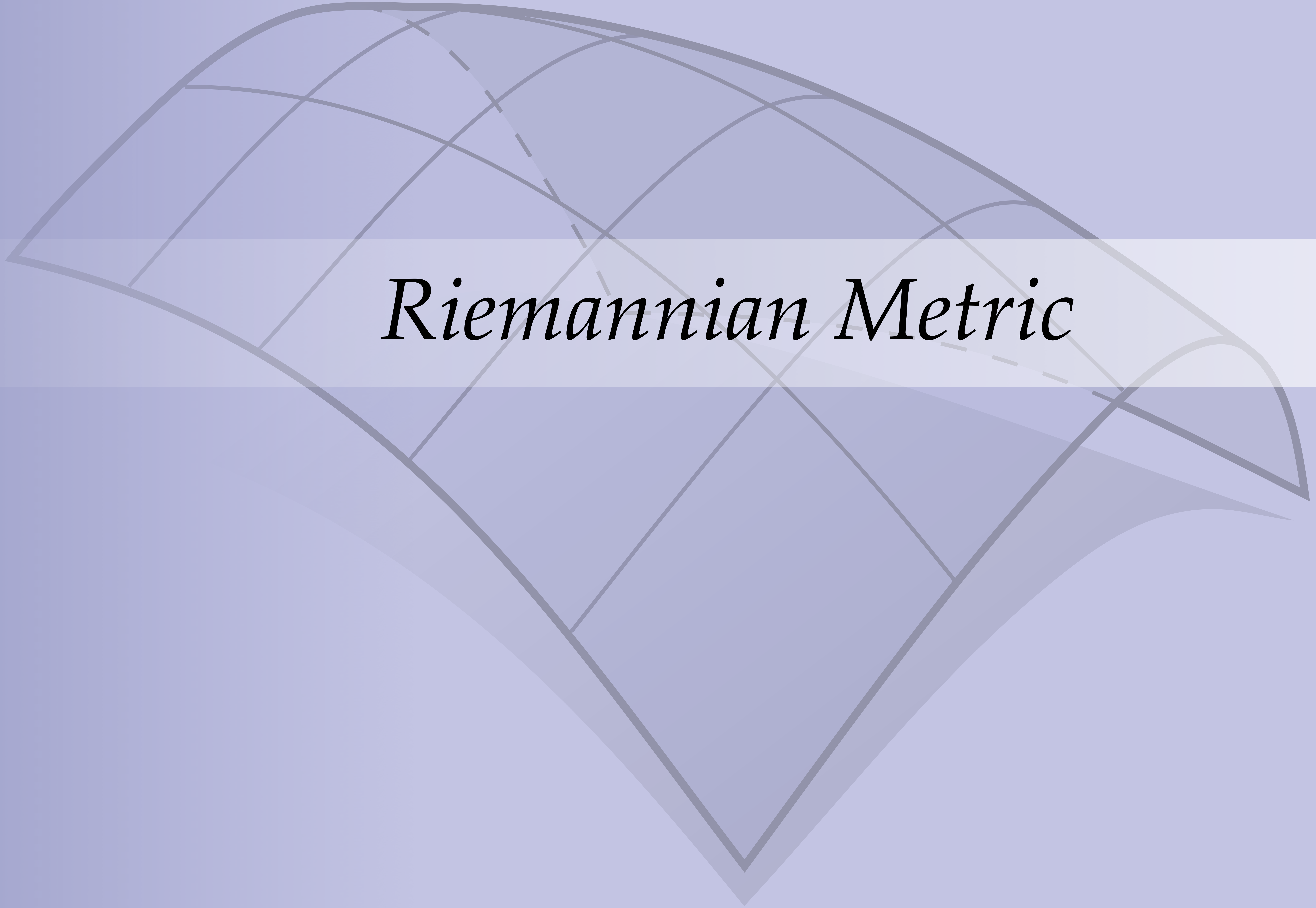
Sphere Eversion

Turning a Sphere Inside-Out (1994)



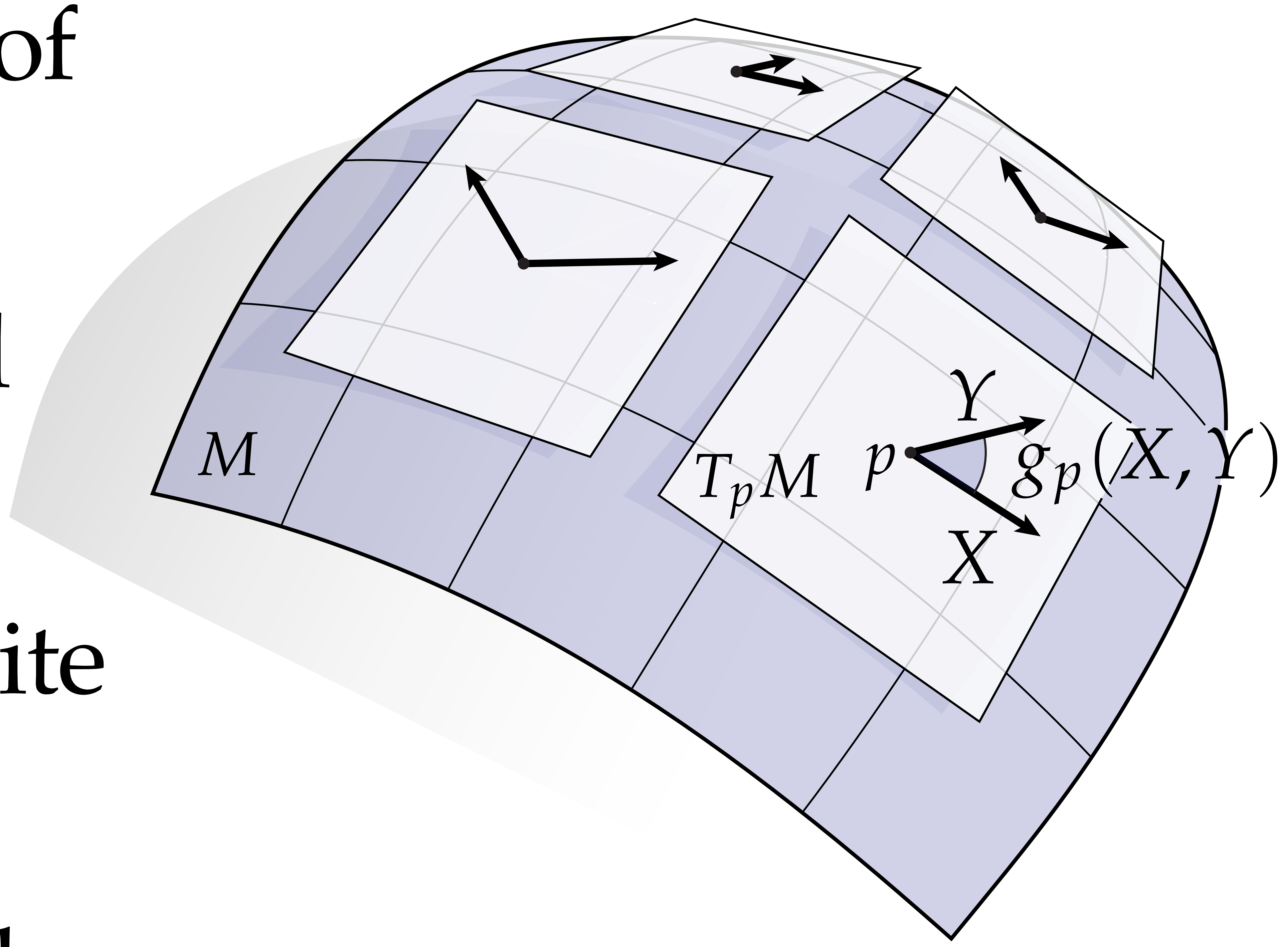
https://youtu.be/-6g3ZcmjJ7k

Riemannian Metric



Riemann Metric

- Many quantities on manifolds (curves, surfaces, *etc.*) ultimately boil down to measurements of *lengths* and *angles* of tangent vectors
- This information is encoded by the so-called *Riemannian metric**
- Abstractly: smoothly-varying positive-definite bilinear form
- For immersed surface, can (and will!) describe more concretely/geometrically

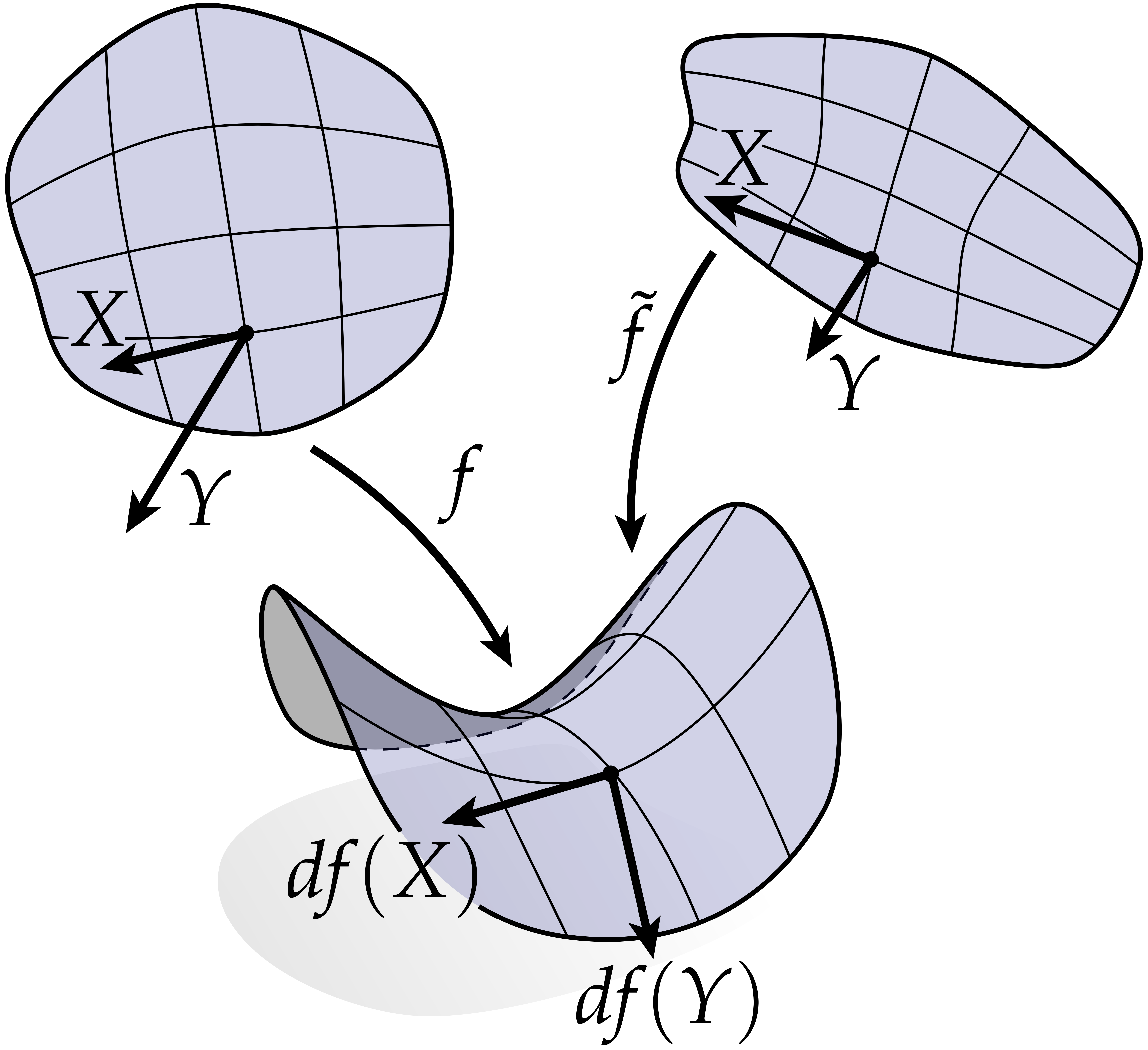


*__Note:__ *not* the same as a point-to-point distance metric $d(x,y)$

Metric Induced by an Immersion

- Given an immersed surface f, how should we measure inner product of vectors X, Y on its domain U?
- We should **not** use the usual inner product on the plane! (Why not?)
- Planar inner product tells us *nothing* about actual length & angle on the surface (and changes depending on choice of parameterization!)
- Instead, use **induced metric**

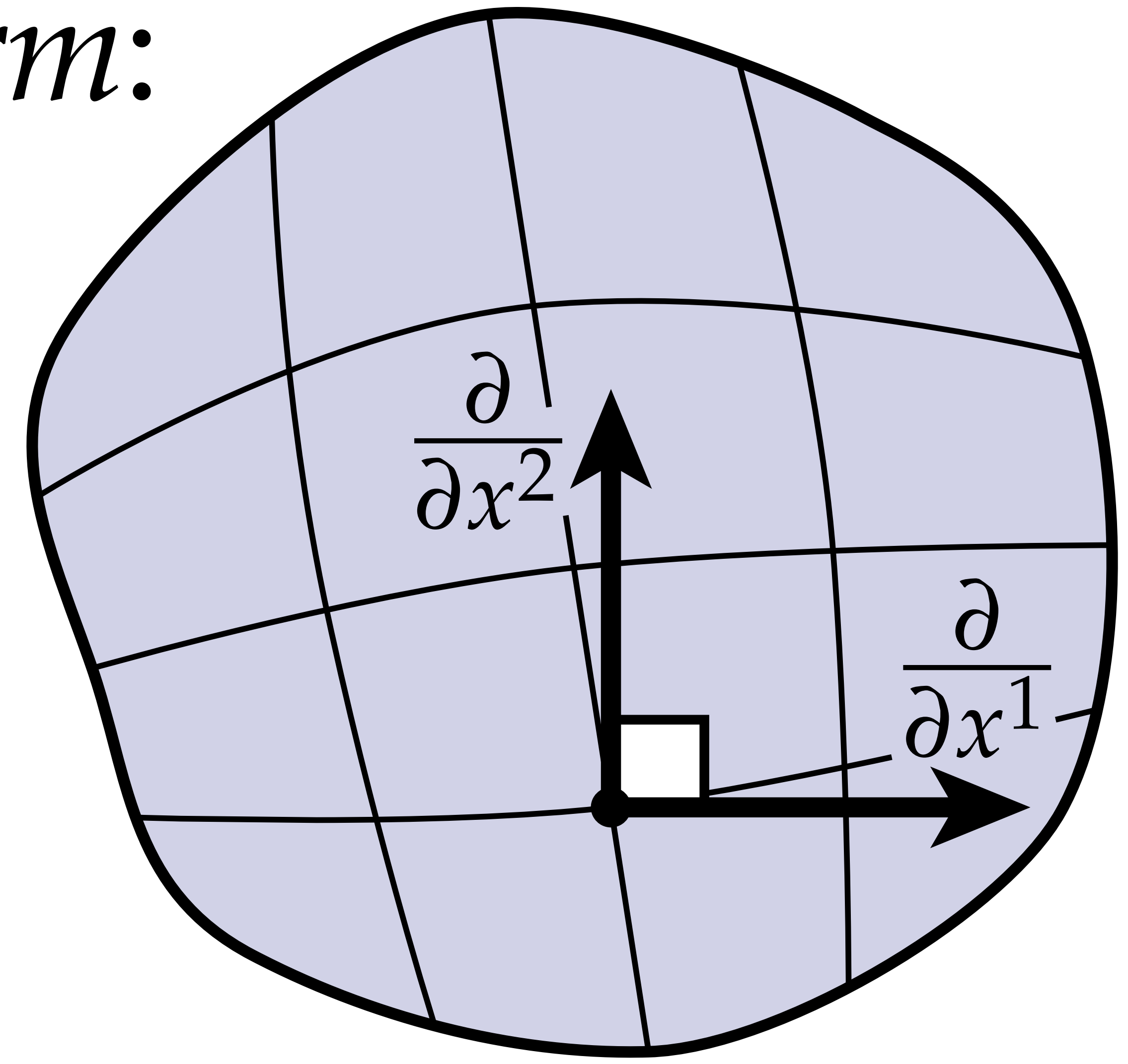
$$g(X,Y) := \langle df(X), df(Y) \rangle$$



Key idea: must account for "stretching"

Induced Metric—Matrix Representation

- Metric is a bilinear map from a pair of vectors to a scalar, which we can represent as a 2x2 matrix $\mathbf{I}$ called the *first fundamental form*:



$$g(X,Y) = X^T \mathbf{I} Y$$

$$\Rightarrow \mathbf{I}_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle df\left(\frac{\partial}{\partial x^i}\right), df\left(\frac{\partial}{\partial x^j}\right)\right\rangle$$

- Alternatively, can express first fundamental form via Jacobian:

$$g(X,Y) = \langle df(X), df(Y)\rangle = (J_f X)^\mathsf{T}(J_f Y) = X^\mathsf{T}(J_f^\mathsf{T} J_f)Y$$

$$\Rightarrow \mathbf{I} = J_f^\mathsf{T} J_f$$

Induced Metric—Example

Can use the differential to obtain the induced metric:

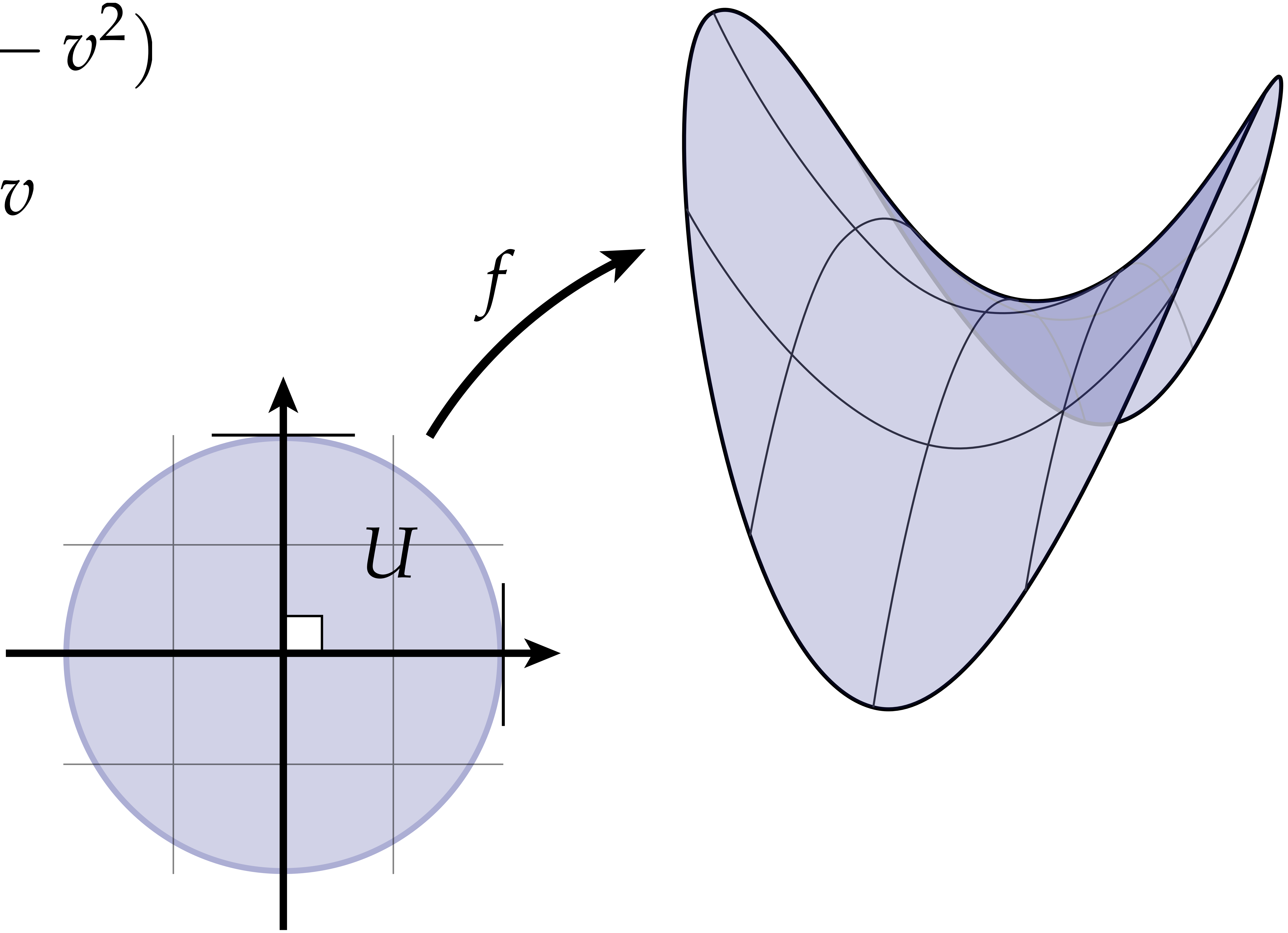
$$f : U \to \mathbb{R}^3;\ (u, v) \mapsto (u, v, u^2 - v^2)$$

$$df = (1, 0, 2u)du + (0, 1, -2v)dv$$

$$J_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

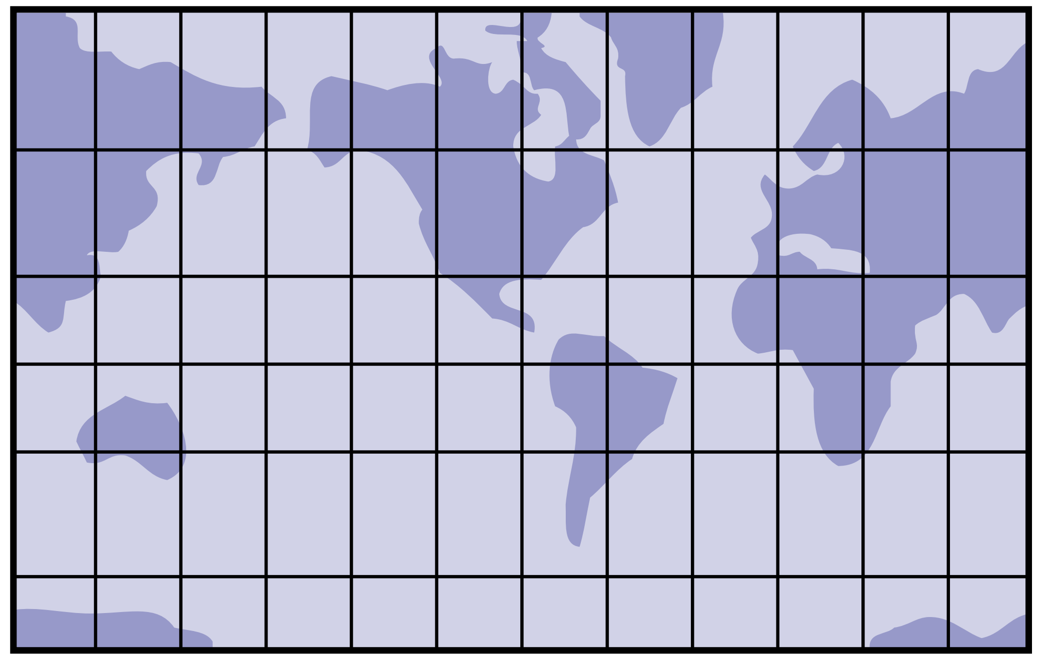
$$\mathbf{I} = J_f^\mathsf{T} J_f$$

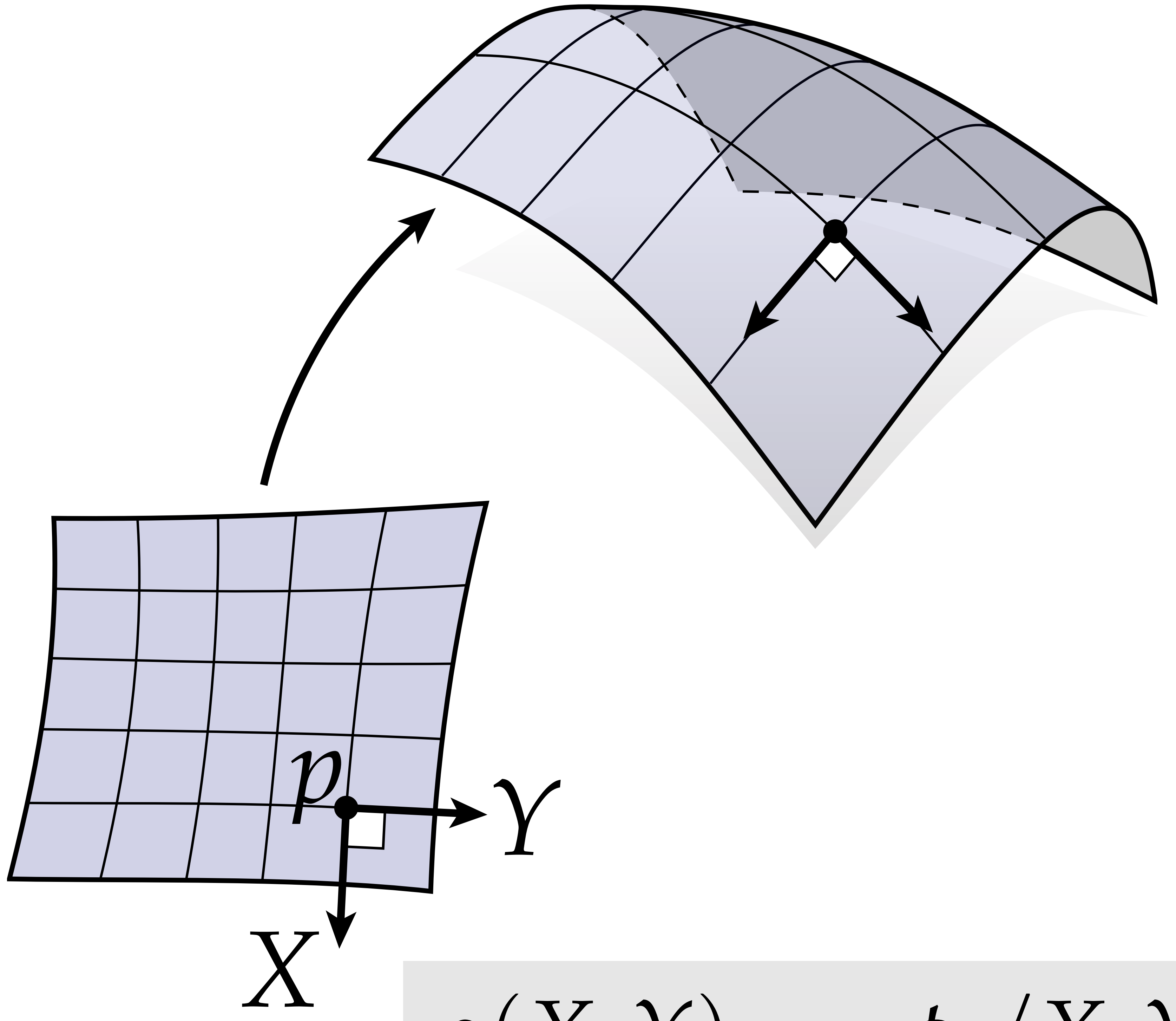
$$= \begin{bmatrix} 1 + 4u^2 & -4uv \\ -4uv & 1 + 4v^2 \end{bmatrix}$$



Conformal Coordinates

- As we've just seen, there can be a complicated relationship between length & angle on the domain (2D) and the image (3D)
- For curves, we simplified life by using an *arc-length* or *isometric* parameterization: lengths on domain are identical to lengths along curve
- For surfaces, usually not possible to preserve all *lengths* (e.g., globe). Remarkably, however, can always preserve *angles* (**conformal**)
- Equivalently, a parameterized surface is *conformal* if at each point the induced metric is simply a positive rescaling of the 2D Euclidean metric





$$g(X,Y)_p = \phi_p \langle X, Y \rangle$$

Example (Enneper Surface)

Consider the surface

$$f(u,v) := \begin{bmatrix} uv^2 + u - \frac{1}{3}u^3 \\ \frac{1}{3}v\left(v^2 - 3u^2 - 3\right) \\ (u-v)(u+v) \end{bmatrix}$$

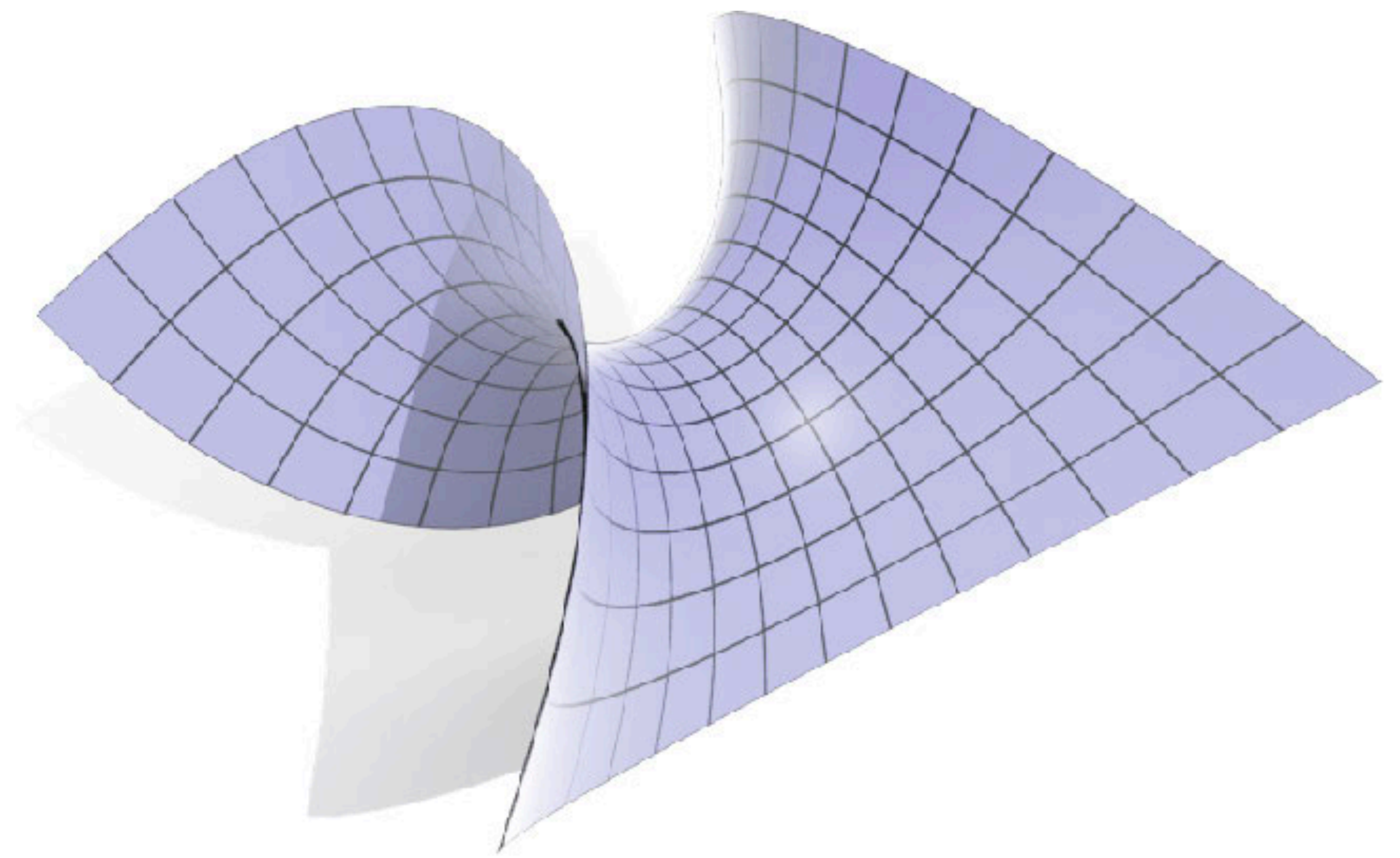
Its Jacobian matrix is

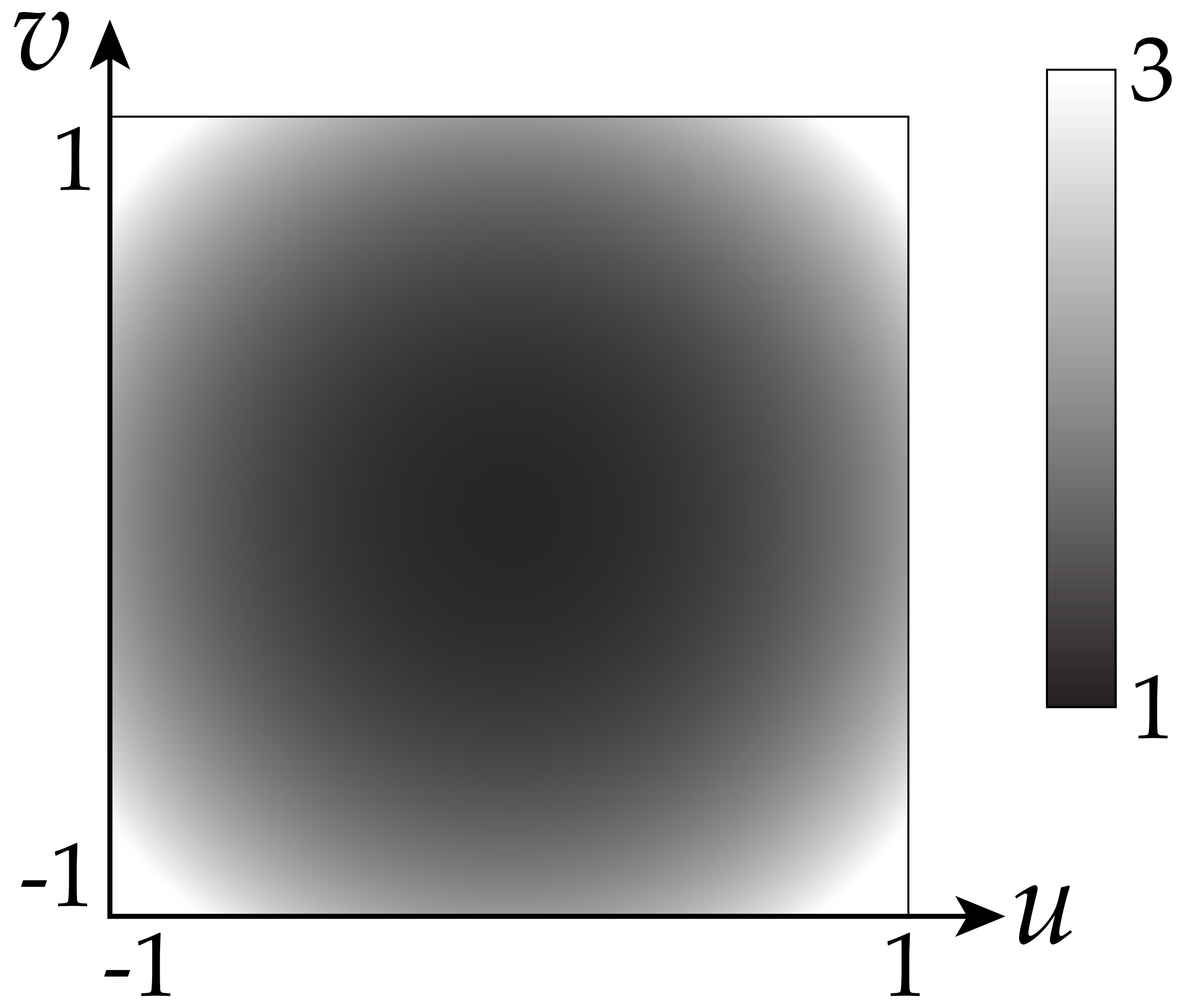
$$J_f = \begin{bmatrix} -u^2+v^2+1 & 2uv \\ -2uv & -u^2+v^2-1 \\ 2u & -2v \end{bmatrix}$$

Its metric then works out to be just a scalar function times the usual metric of the Euclidean plane:

$$\mathbf{I} = J_f^T J_f = \left(u^2+v^2+1\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This function is called the *conformal scale factor*.





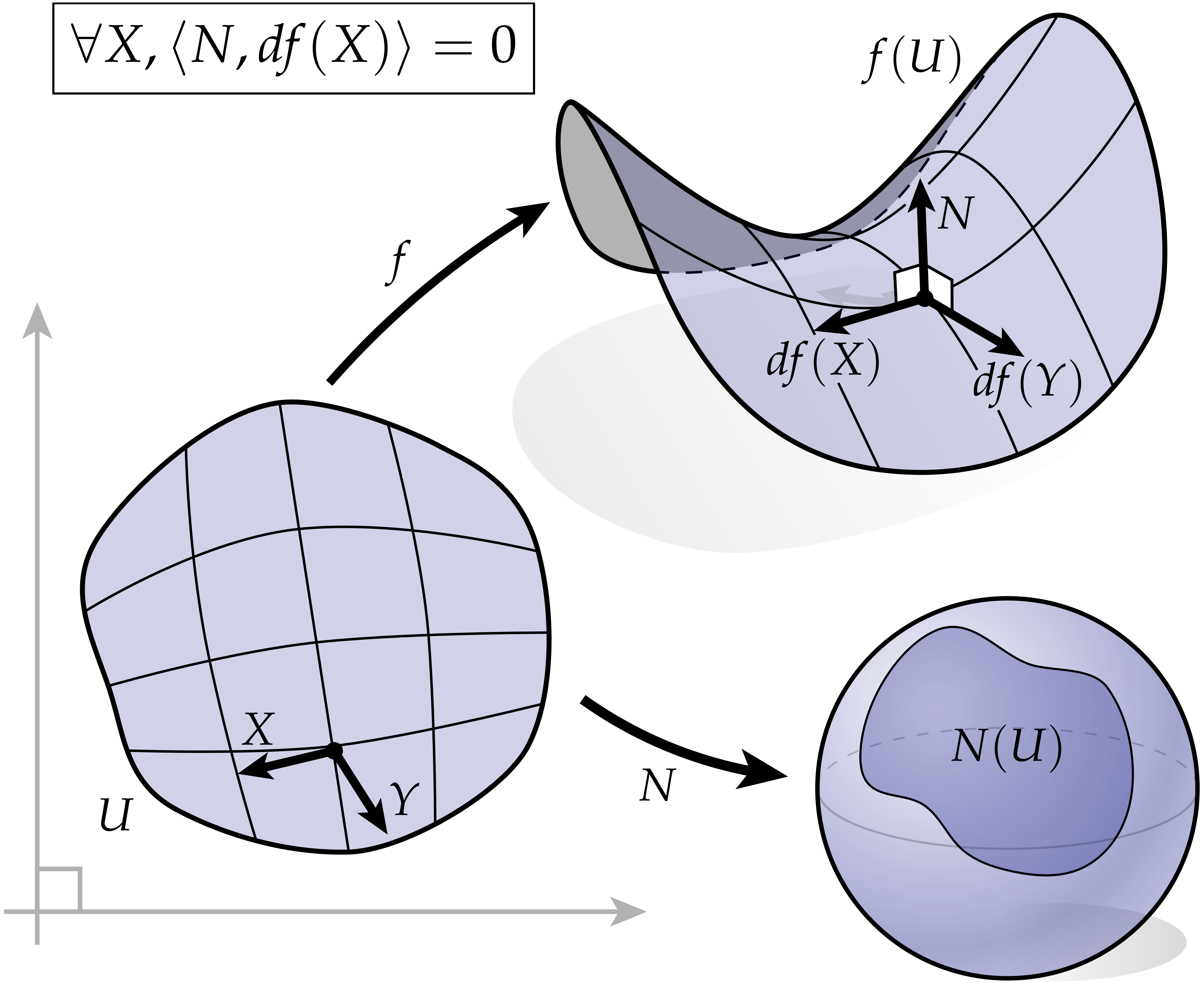
conformal factor

Gauss Map

Gauss Map

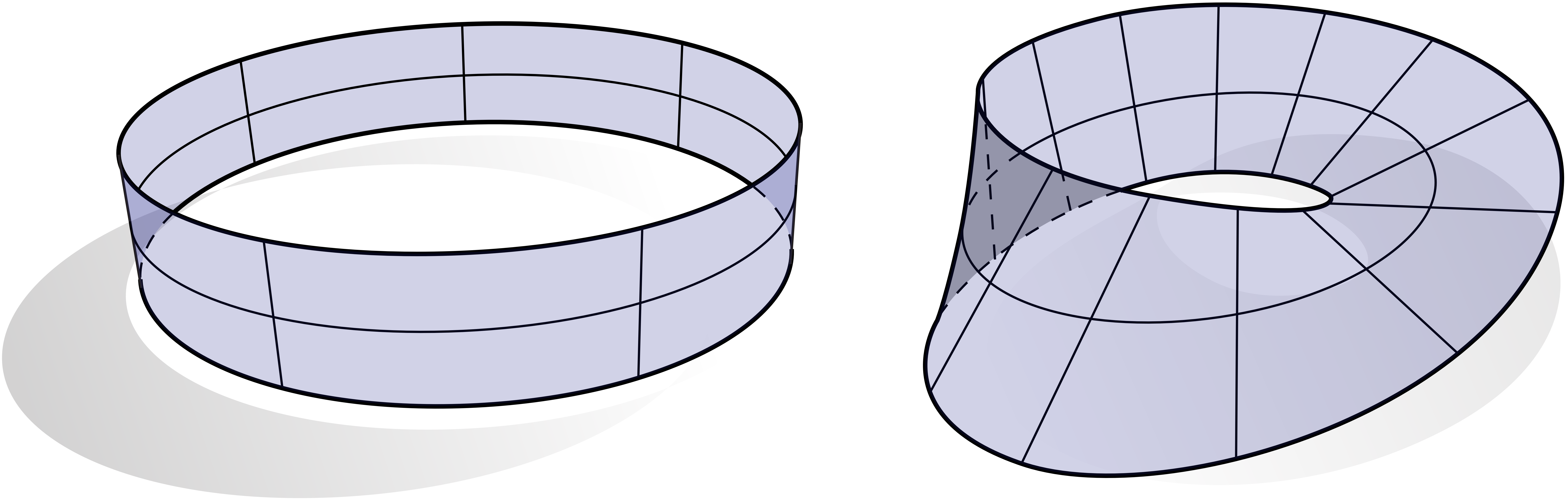
- A vector is **normal** to a surface if it is orthogonal to all tangent vectors
- **Q:** Is there a *unique* normal at a given point?
- **A:** No! Can have different magnitudes/directions.
- The **Gauss map** is a *continuous* map taking each point on the surface to a *unit* normal vector
- Can visualize Gauss map as a map from the surface to the unit sphere

$$\boxed{\forall X, \langle N, df(X) \rangle = 0}$$



Orientability

Not every surface admits a Gauss map (globally):



orientable **nonorientable**

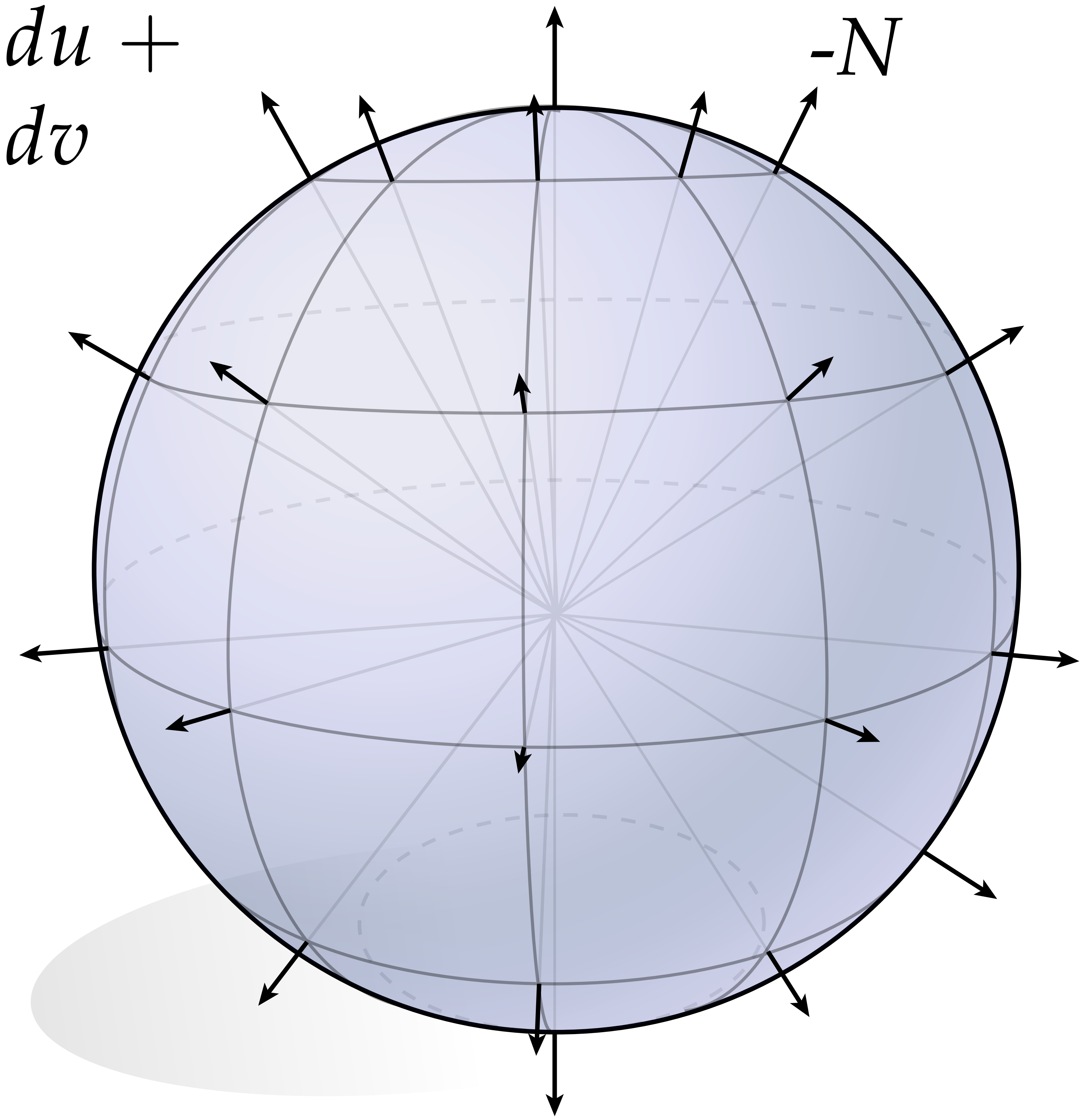
Gauss Map—Example

Can obtain unit normal by taking the cross product of two tangents*:

$$f := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

$$df = \begin{array}{lcccl} (& -\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &)\ du\ + \\ (& \cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &)\ dv \end{array}$$

$$df(\tfrac{\partial}{\partial u}) \times df(\tfrac{\partial}{\partial v}) = \begin{bmatrix} -\cos(u)\sin^2(v) \\ -\sin(u)\sin^2(v) \\ -\cos(v)\sin(v) \end{bmatrix}$$



To get *unit* normal, divide by length. In this case, can just notice we have a constant multiple of the sphere itself:

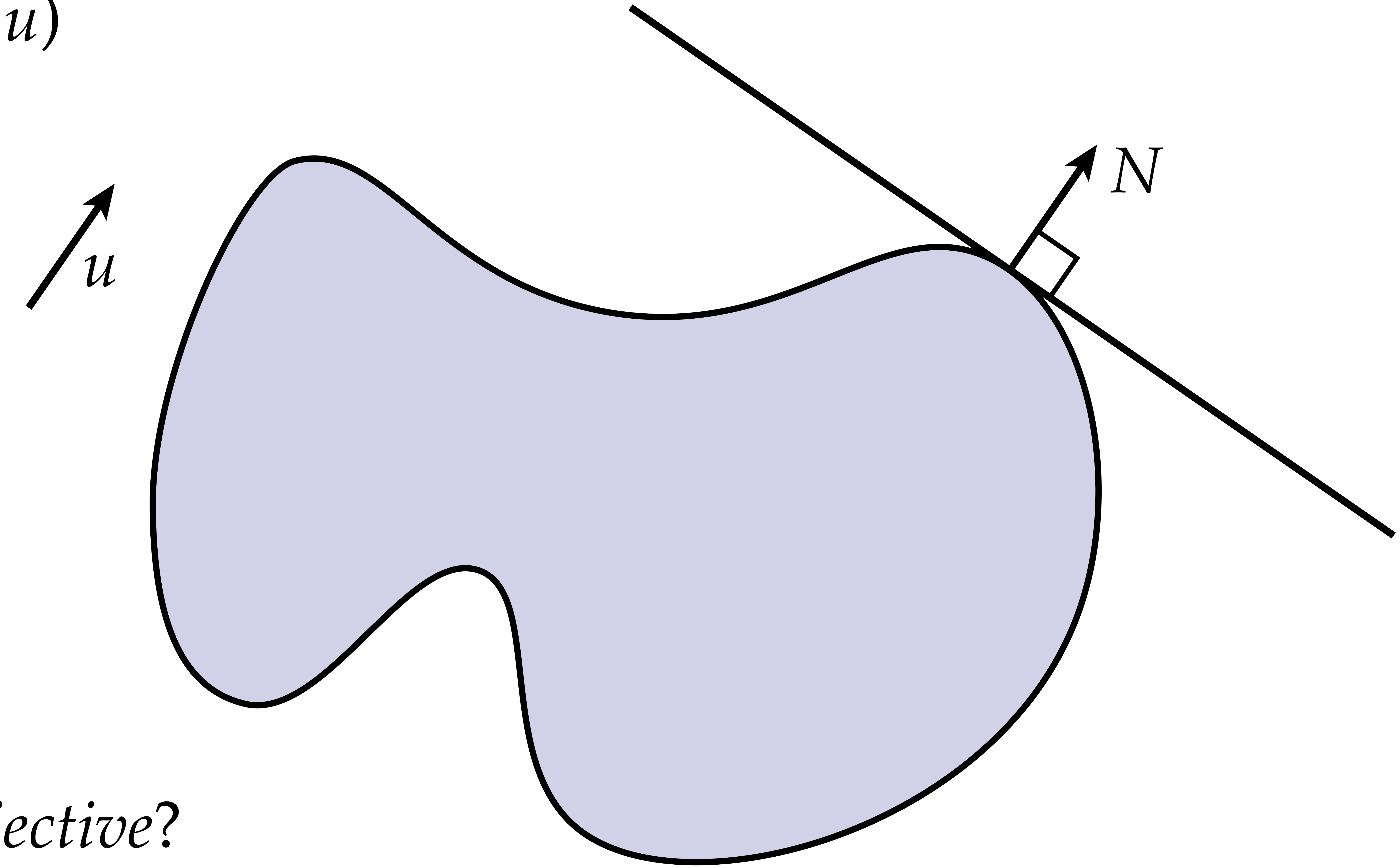
$$\Rightarrow N = -f$$

*Must not be parallel!

Surjectivity of Gauss Map

- Given a unit vector u, can we always find some point on a surface that has this normal? ($N = u$)

- Yes! **Proof** (Hilbert):



Q: Is the Gauss map *injective*?

Vector Area

- Given a little patch of surface Ω, what's the "average normal"?
- Can simply integrate normal over the patch, divide by area:

$$\frac{1}{\text{area}(\Omega)} \int_\Omega N \, dA$$

- Integrand $N\,dA$ is called the **vector area**. (Vector-valued 2-form)
- Can be easily expressed via exterior calculus*:

$$\begin{aligned} df \wedge df(X,Y) = {} & df(X) \times df(Y) - df(Y) \times df(X) = \\ & 2df(X) \times df(Y) = \\ & 2NdA(X,Y) \end{aligned}$$

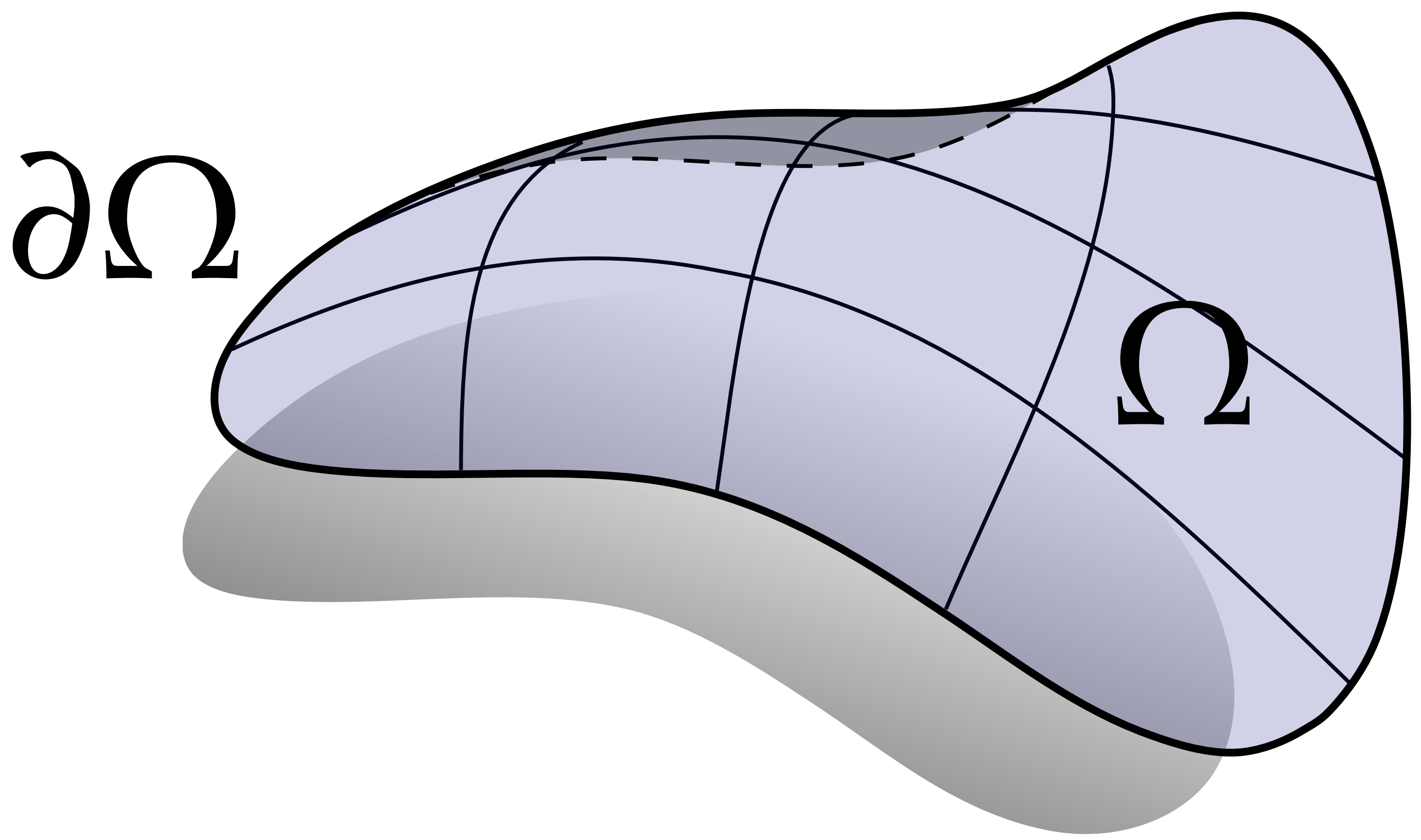
$$\implies \boxed{\mathcal{A} = \tfrac{1}{2} df \wedge df}$$

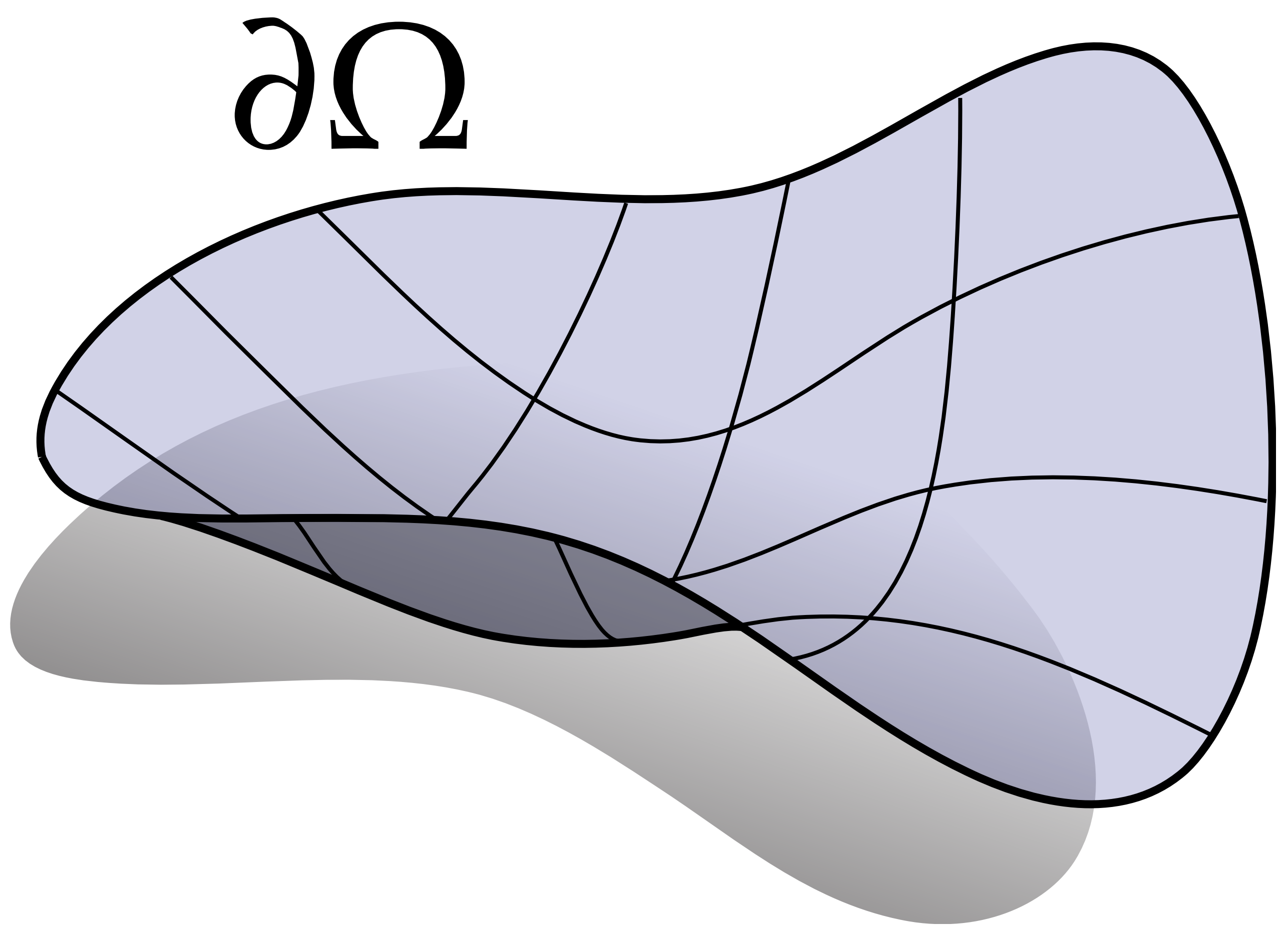
Vector Area, continued

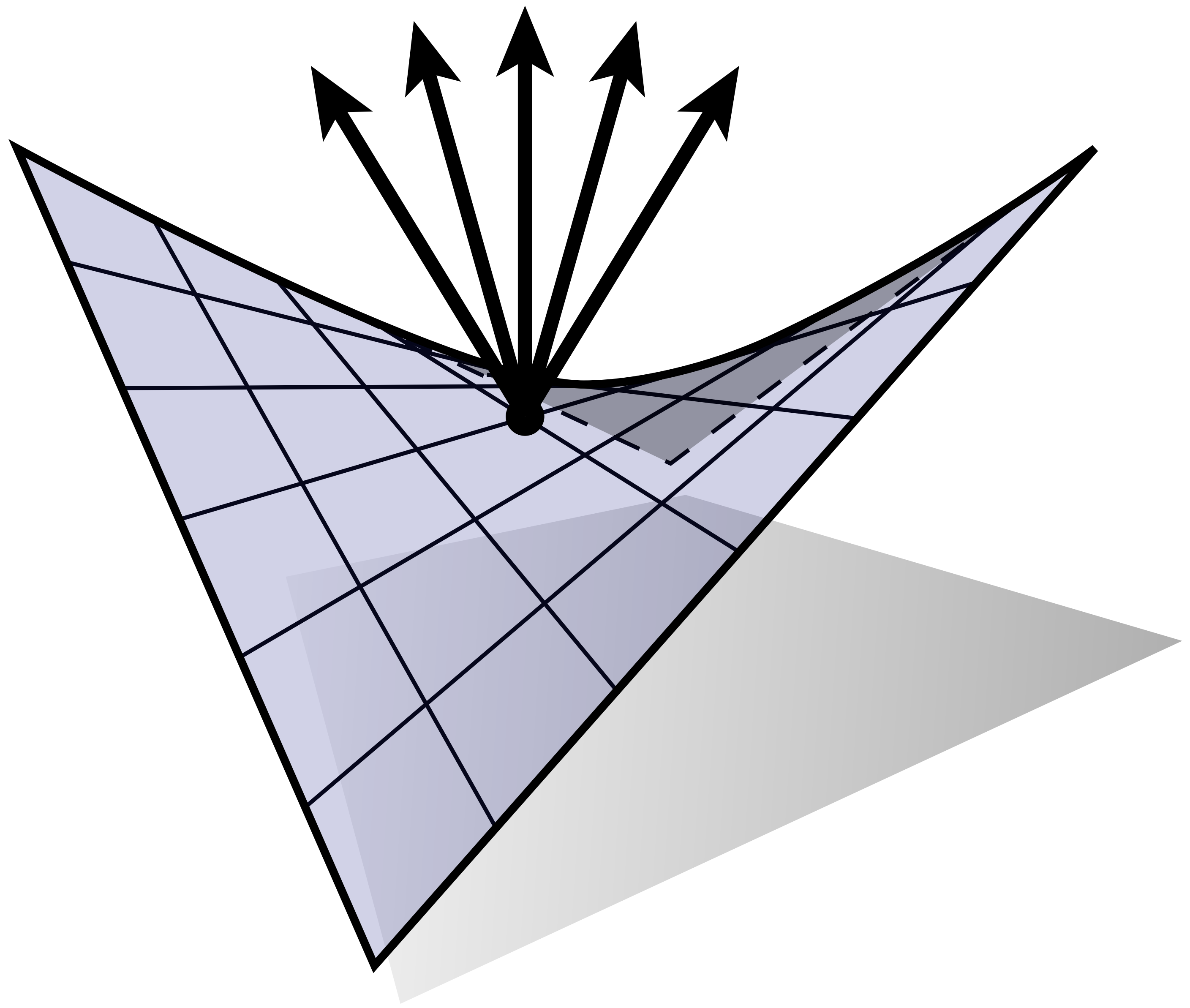
- By expressing vector area this way, we make an interesting observation:

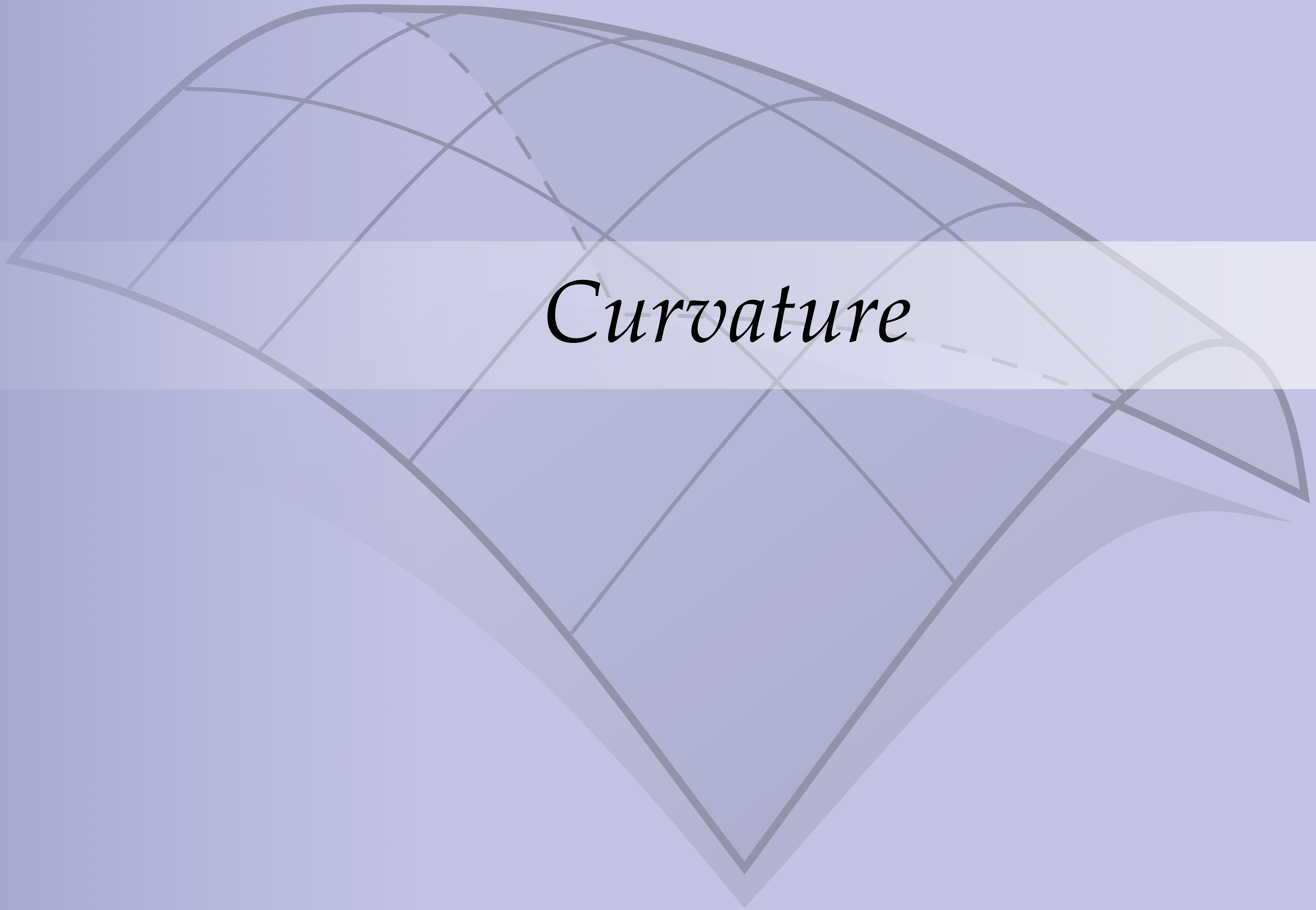
$$2\int_\Omega N\,dA = \int_\Omega df \wedge df = \int_\Omega d(f\,df) = \int_{\partial\Omega} f\,df = \int_{\partial\Omega} f(s) \times df(T(s))\,ds$$

- Hence, vector area is the same for any two patches w/ same boundary
- Can define "normal" given **only** boundary (*e.g.*, nonplanar polygon)
- **Corollary:** *integral of normal vanishes for any closed surface*





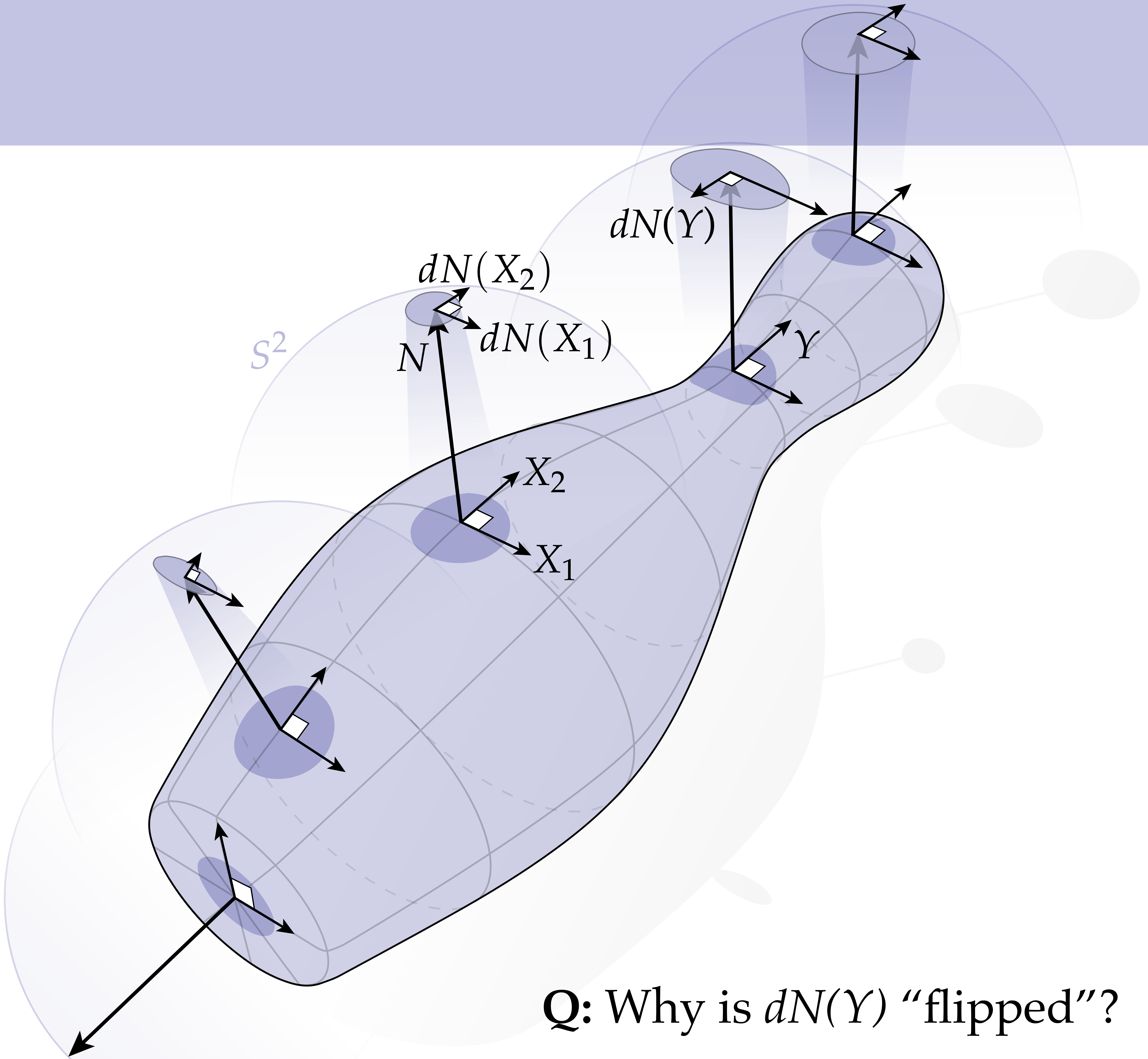




Curvature

Weingarten Map

- The **Weingarten map** dN is the differential of the Gauss map N
- At each point, tells us the change in the normal vector along any given direction X
- Since change in *unit* normal cannot have any component in the normal direction, $dN(X)$ is always tangent to the surface
- Can also think of it as a vector tangent to the unit sphere S^2



Q: Why is $dN(Y)$ "flipped"?

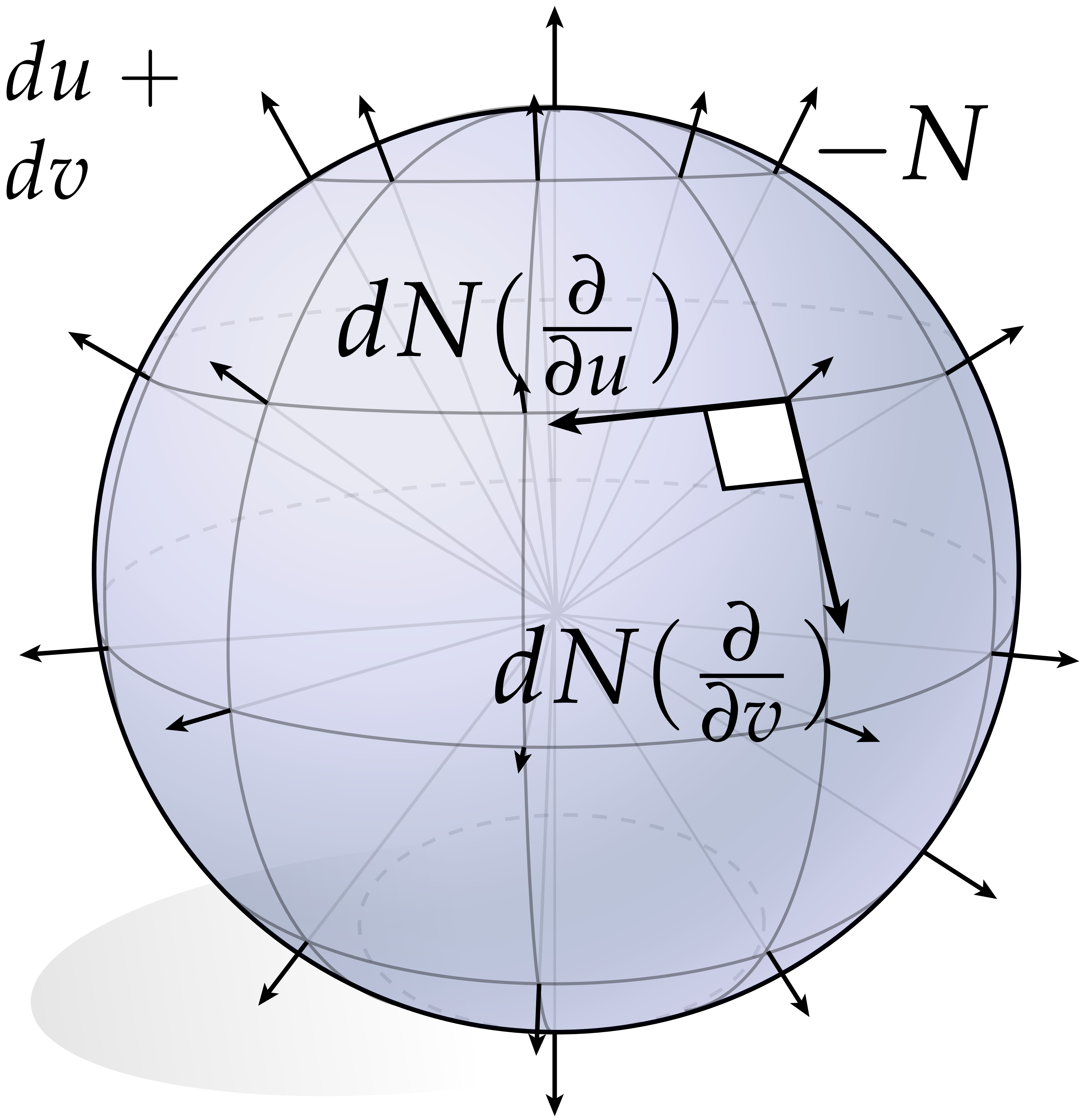
Weingarten Map—Example

- Recall that for the sphere, $N = -f$. Hence, Weingarten map dN is just $-df$:

$$f := (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

$$df = \begin{matrix} (& -\sin(u)\sin(v), & \cos(u)\sin(v), & 0 &) & du\ + \\ (& \cos(u)\cos(v), & \cos(v)\sin(u), & -\sin(v) &) & dv \end{matrix}$$

$$dN = \begin{matrix} (& \sin(u)\sin(v), & -\cos(u)\sin(v), & 0 &) & du \\ (& -\cos(u)\cos(v), & -\cos(v)\sin(u), & \sin(v) &) & dv \end{matrix}$$



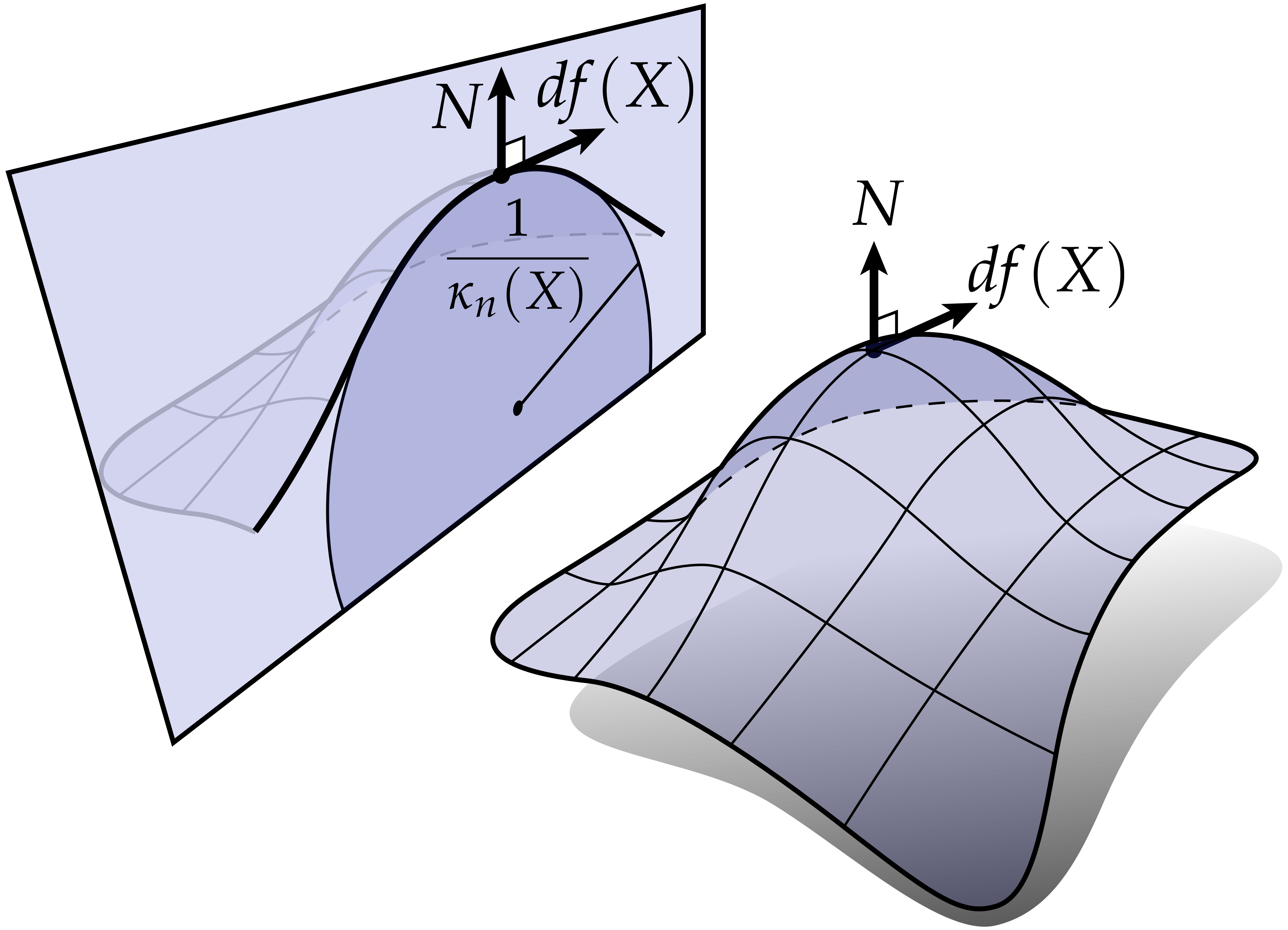
Key idea: computing the Weingarten map is no different from computing the differential of a surface.

Normal Curvature

- For curves, curvature was the rate of change of the *tangent*; for immersed surfaces, we'll instead consider how quickly the *normal* is changing.*

- In particular, **normal curvature** is rate at which normal is bending along a given tangent direction:

$$\kappa_N(X) := \frac{\langle df(X), dN(X)\rangle}{|df(X)|^2}$$



- Equivalent to intersecting surface with normal-tangent plane and measuring the usual curvature of a plane curve

*For plane curves, what would happen if we instead considered change in N?

Normal Curvature—Example

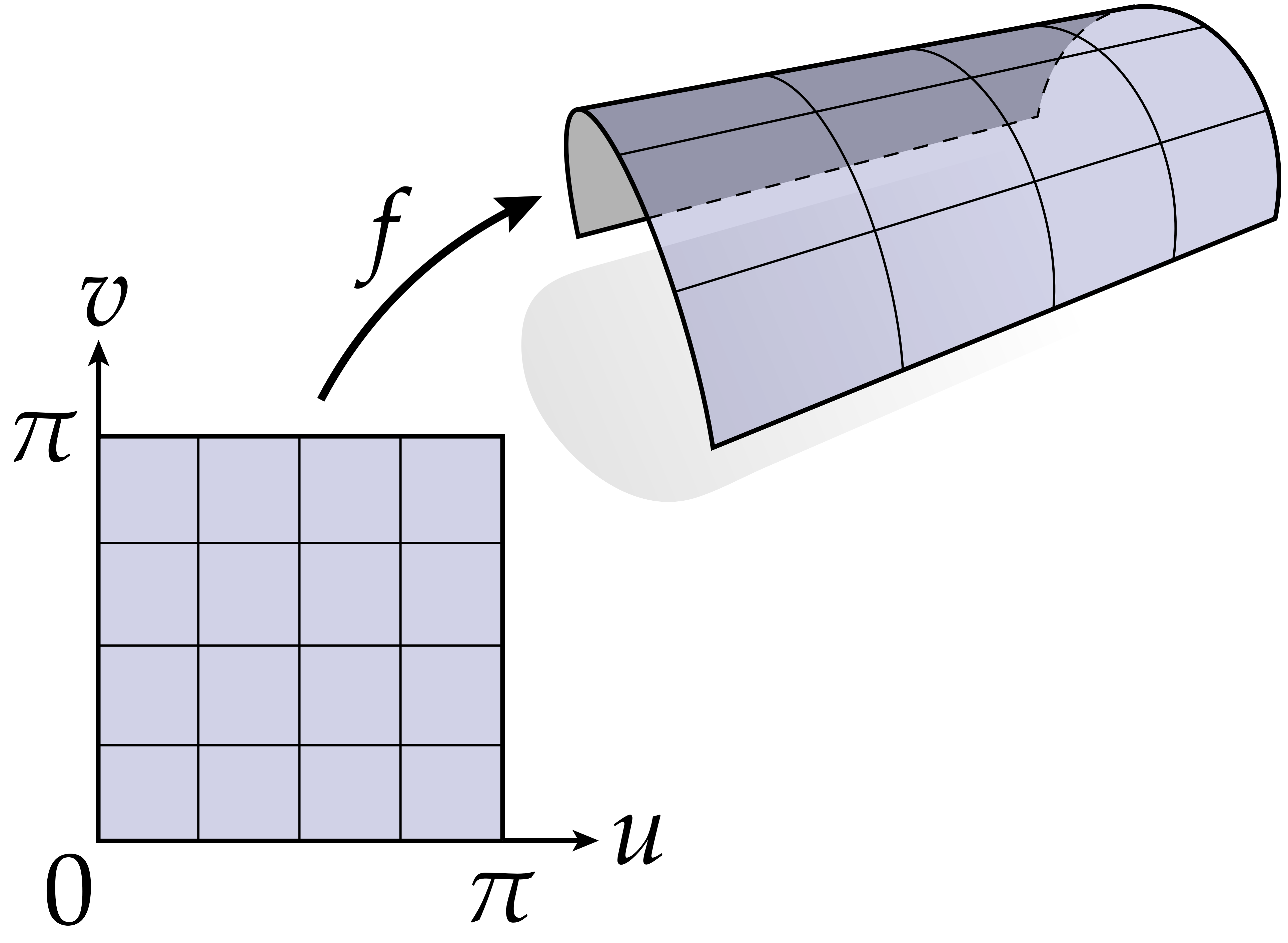
Consider a parameterized cylinder:

$$f(u,v) := (\cos(u), \sin(u), v)$$

$$df = (-\sin(u), \cos(u), 0)du + (0,0,1)dv$$

$$N = (-\sin(u), \cos(u), 0) \times (0,0,1)$$
$$= (\cos(u), \sin(u), 0)$$

$$dN = (-\sin(u), \cos(u), 0)du$$



$$\kappa_N(\tfrac{\partial}{\partial u}) = \frac{\langle df(\frac{\partial}{\partial u}), dN(\frac{\partial}{\partial u})\rangle}{|df(\frac{\partial}{\partial u})|^2} = \frac{(-\sin(u),\cos(u),0)\cdot(-\sin(u),\cos(u),0)}{|(-\sin(u),\cos(u),0)|^2} = 1$$

$$\kappa_N(\tfrac{\partial}{\partial v}) = \cdots = 0$$

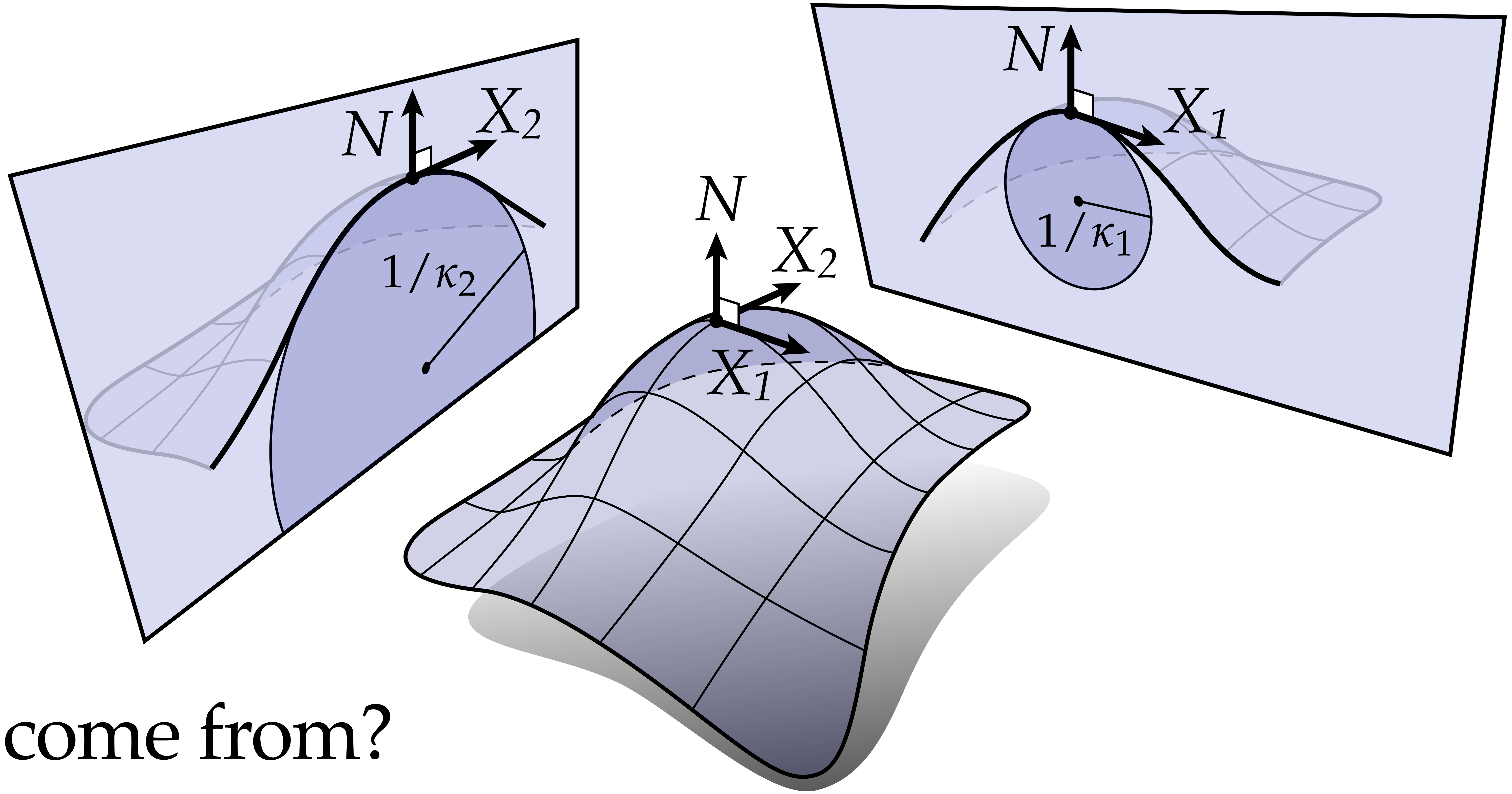
Q: Does this result make sense geometrically?

Principal Curvature

- Among all directions X, there are two **principal directions** X_1, X_2 where normal curvature has minimum/maximum value (respectively)
- Corresponding normal curvatures are the **principal curvatures**
- Two critical facts*:

1. $g(X_1, X_2) = 0$

2. $dN(X_i) = \kappa_i df(X_i)$



Where do these relationships come from?

Shape Operator

- The change in the normal N is always *tangent* to the surface
- Must therefore be some linear map S from tangent vectors to tangent vectors, called the **shape operator**, such that

$$df(SX) = dN(X)$$

- Principal directions are the *eigenvectors* of S
- Principal curvatures are *eigenvalues* of S
- **Note:** S is not a symmetric matrix! Hence, eigenvectors are not orthogonal in R^2; only orthogonal with respect to induced metric g.

Shape Operator — Example

Consider a nonstandard parameterization of the cylinder (*sheared* along z):

$$f(u,v) := (\cos(u), \sin(u), u+v) \qquad df = (-\sin(u), \cos(u), 1)du + (0,0,1)dv$$

$$N = (\cos(u), \sin(u), 0) \qquad dN = (-\sin(u), \cos(u), 0)du$$

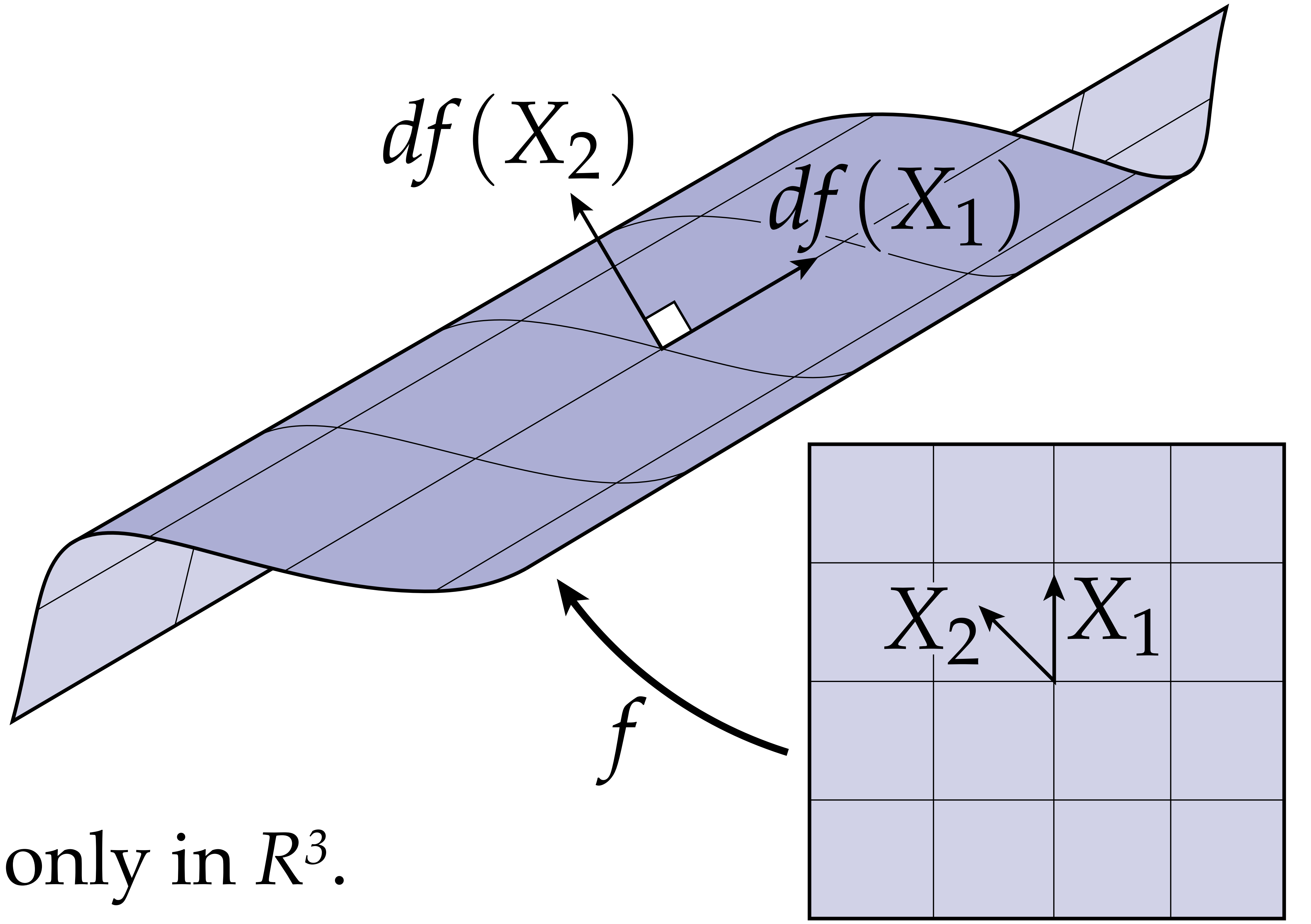
$$\boxed{df \circ S = dN}$$

$$\begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -\sin(u) & 0 \\ \cos(u) & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \qquad X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$df(X_1) = (0,0,1) \qquad \kappa_1 = 0$$

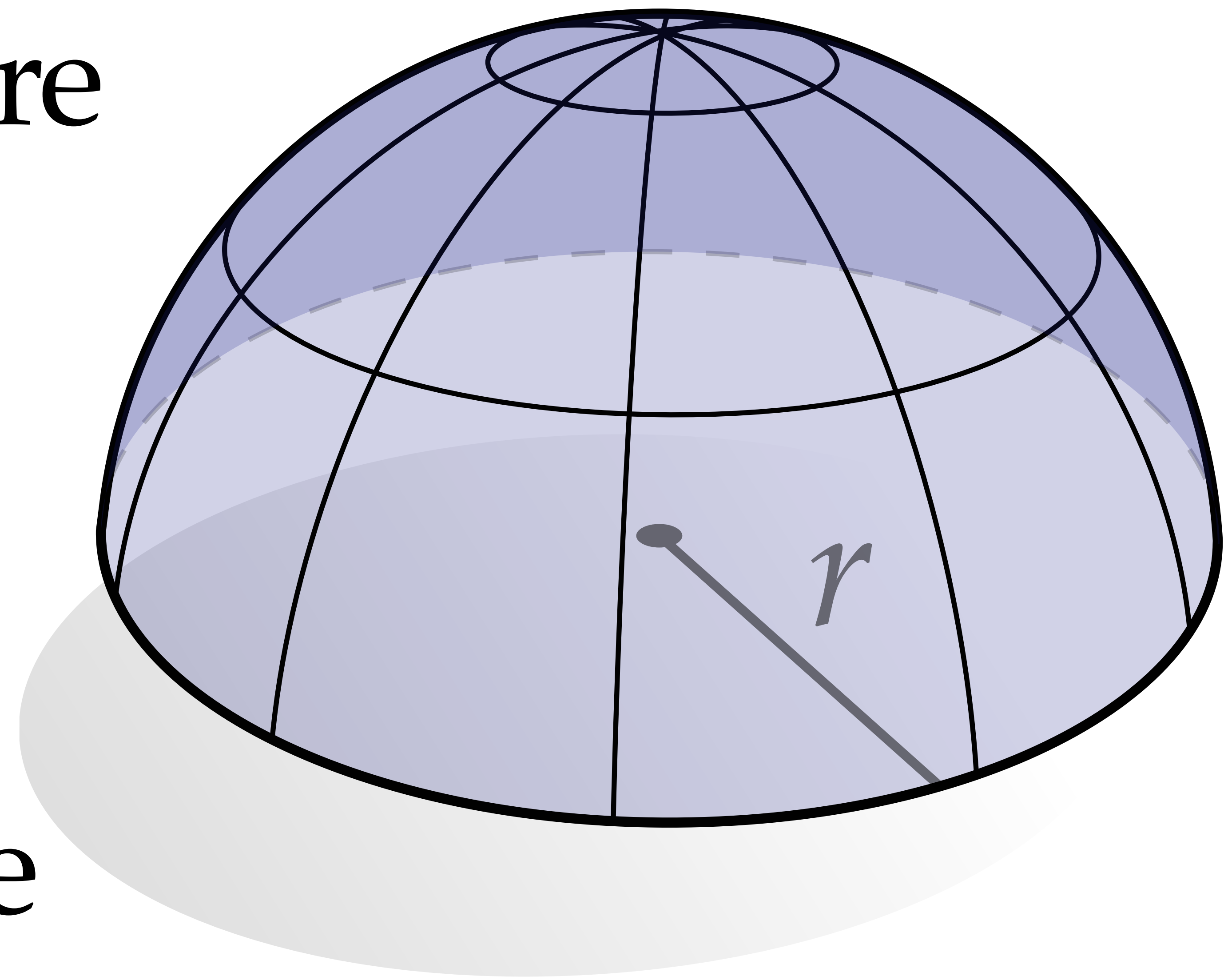
$$df(X_2) = (\sin(u), -\cos(u), 0) \qquad \kappa_2 = 1$$



Key observation: principal directions orthogonal only in R^3.

Umbilic Points

- Points where principal curvatures are equal are called **umbilic points**
- Principal *directions* are not uniquely determined here
- What happens to the shape operator S?
  - May still have full rank!
  - Just have repeated eigenvalues, 2-dim. eigenspace

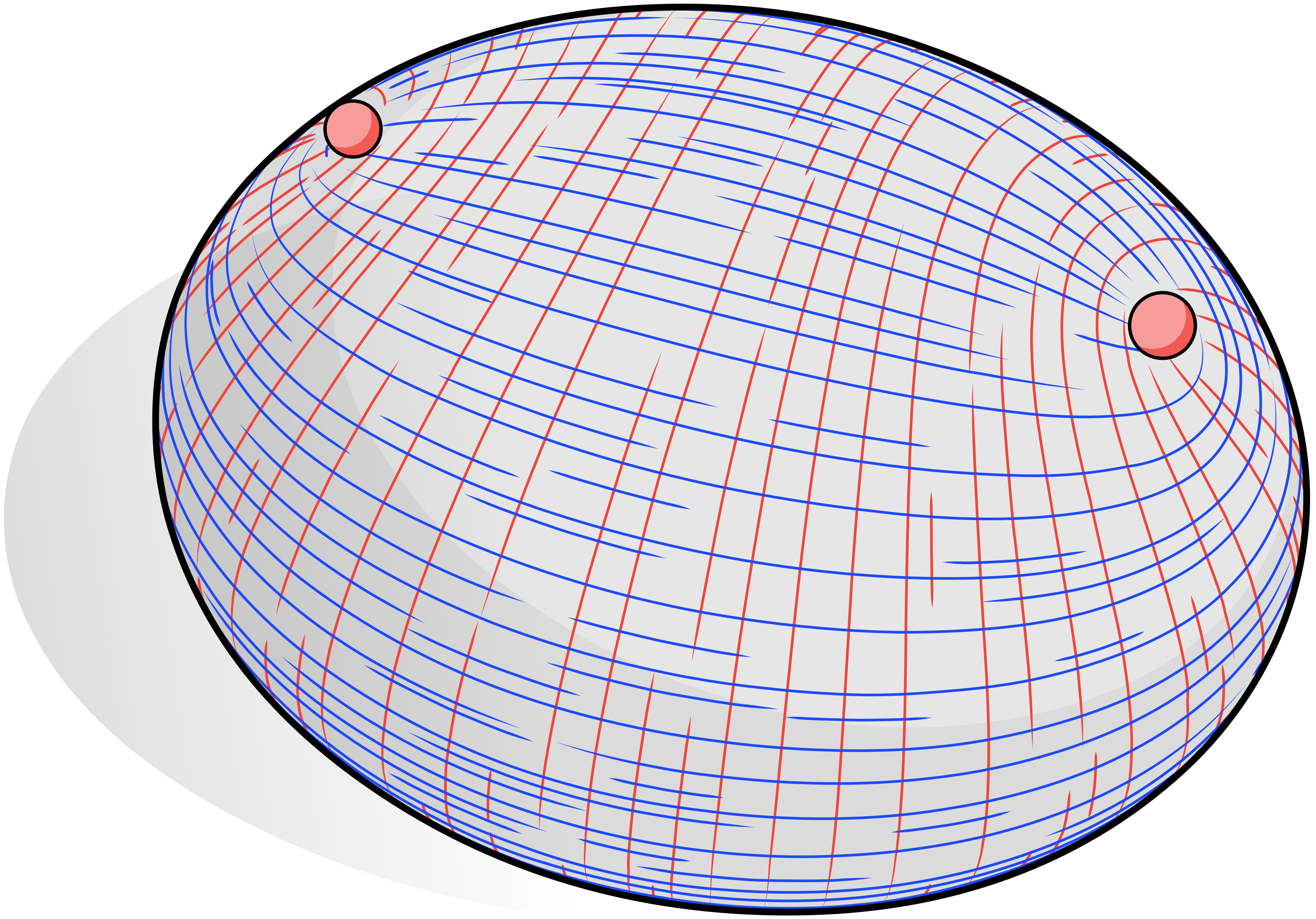


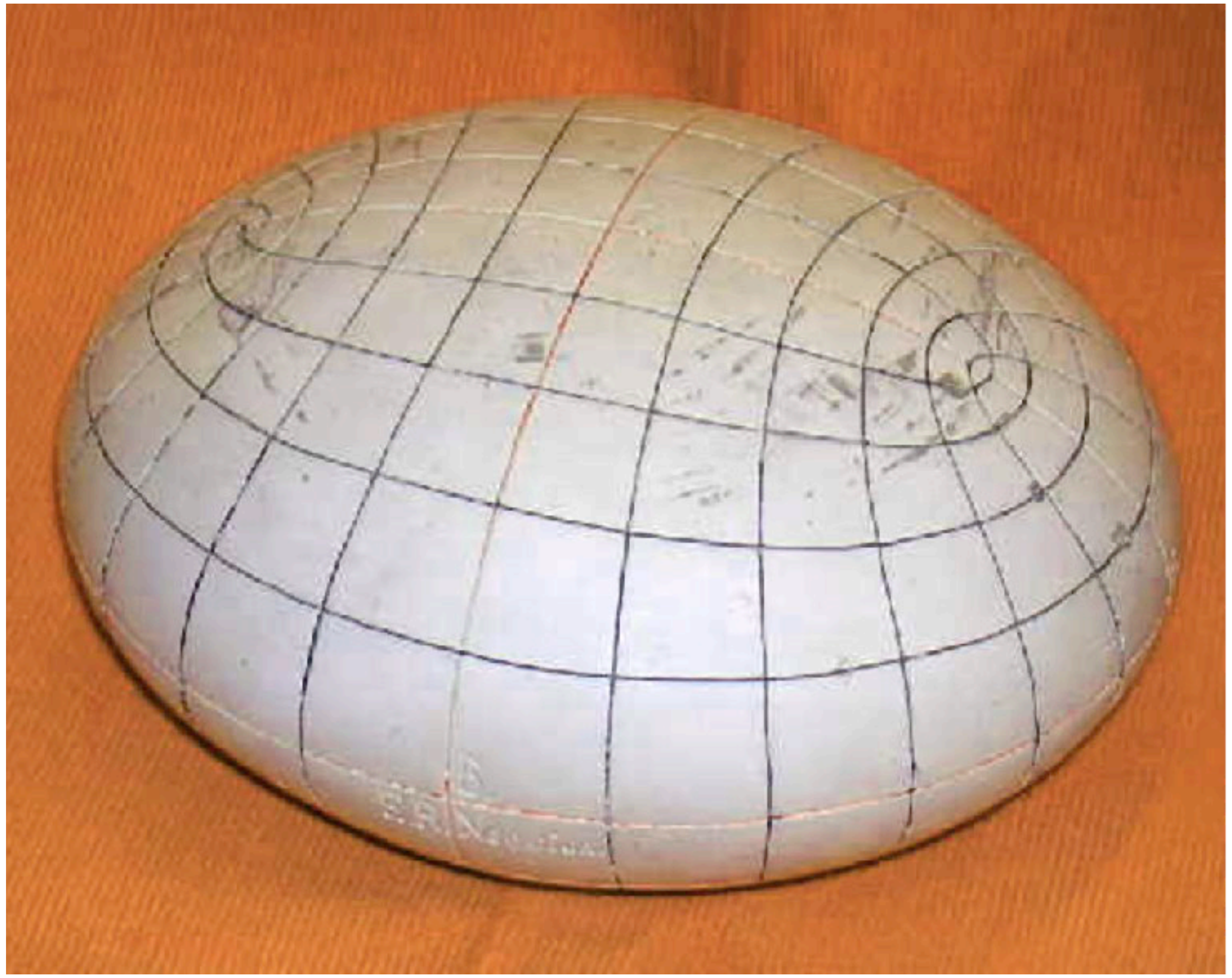
$$S = \begin{bmatrix} 1/r & 0 \\ 0 & 1/r \end{bmatrix} \qquad \kappa_1 = \kappa_2 = \frac{1}{r} \qquad \forall X,\ SX = \tfrac{1}{r}X$$

Could still of course choose (arbitrarily) an orthonormal pair X_1, X_2…

Principal Curvature Nets

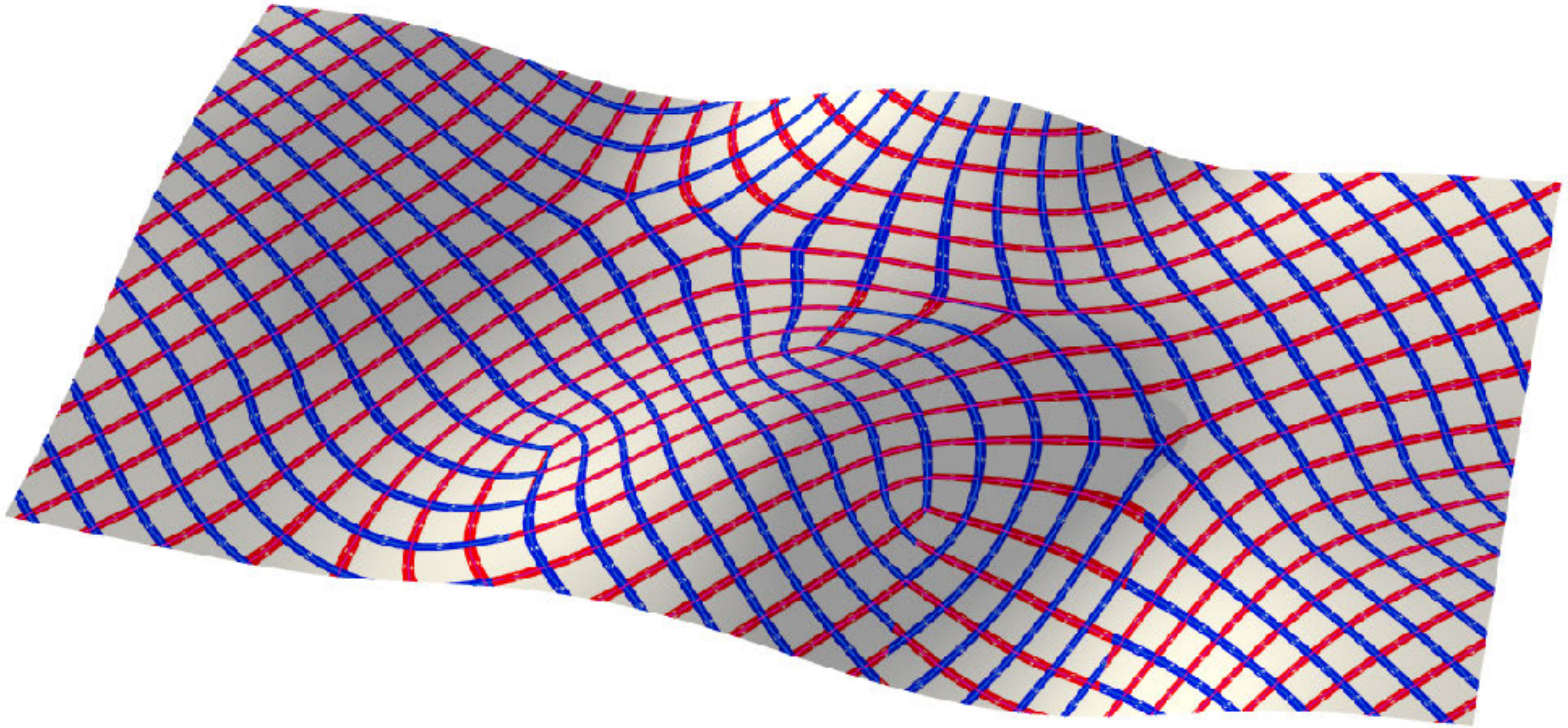
- Walking along principal direction field yields **principal curvature lines**
- Collection of all such lines is called the **principal curvature network**

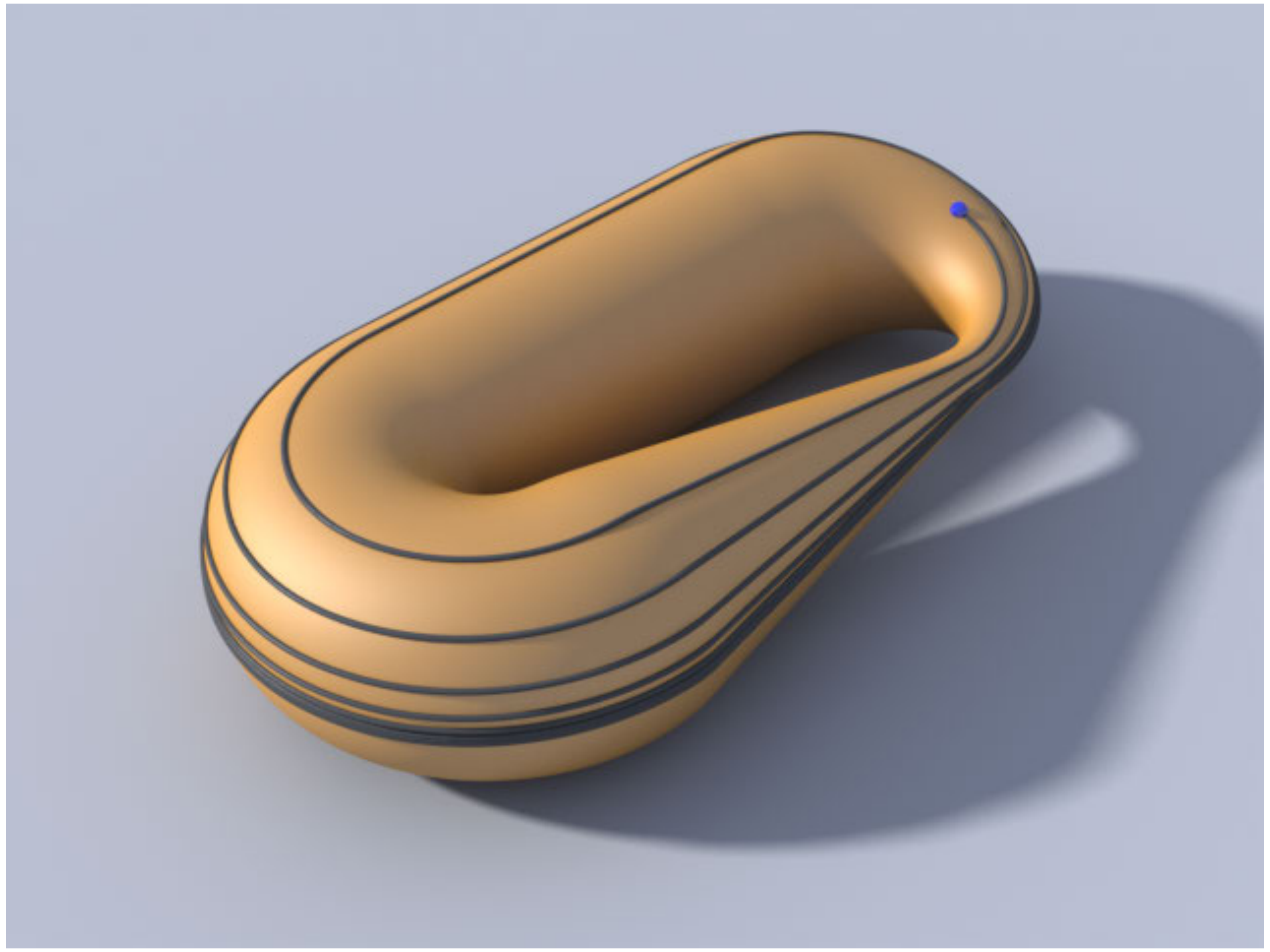




Separatrices and Spirals

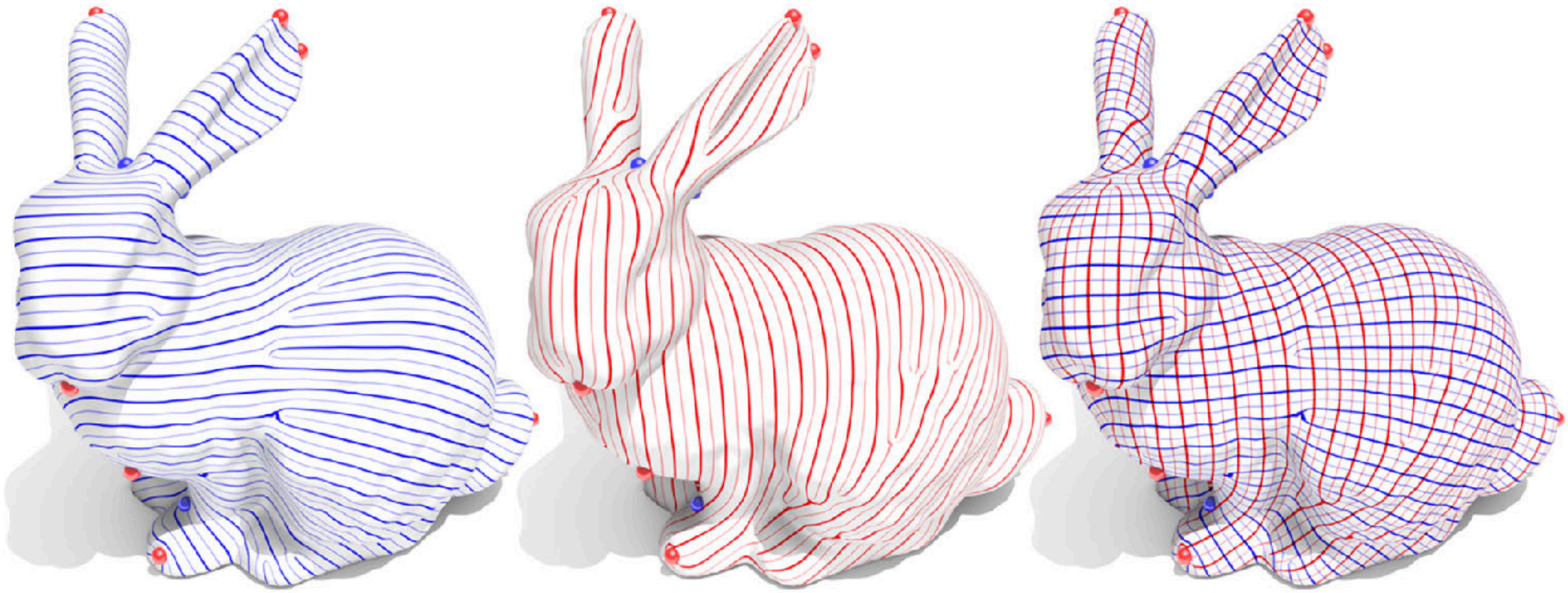
- If we walk along a principal curvature line, where do we end up?
- Sometimes, a curvature line terminates at an umbilic point in both directions; these so-called **separatrices** (can) split network into regular patches.
- Other times, we make a closed loop. More often, however, behavior is *not* so nice!





Application—Quad Remeshing

- Recent approach to meshing: construct net *roughly* aligned with principal curvature—but with separatrices & loops, not spirals.



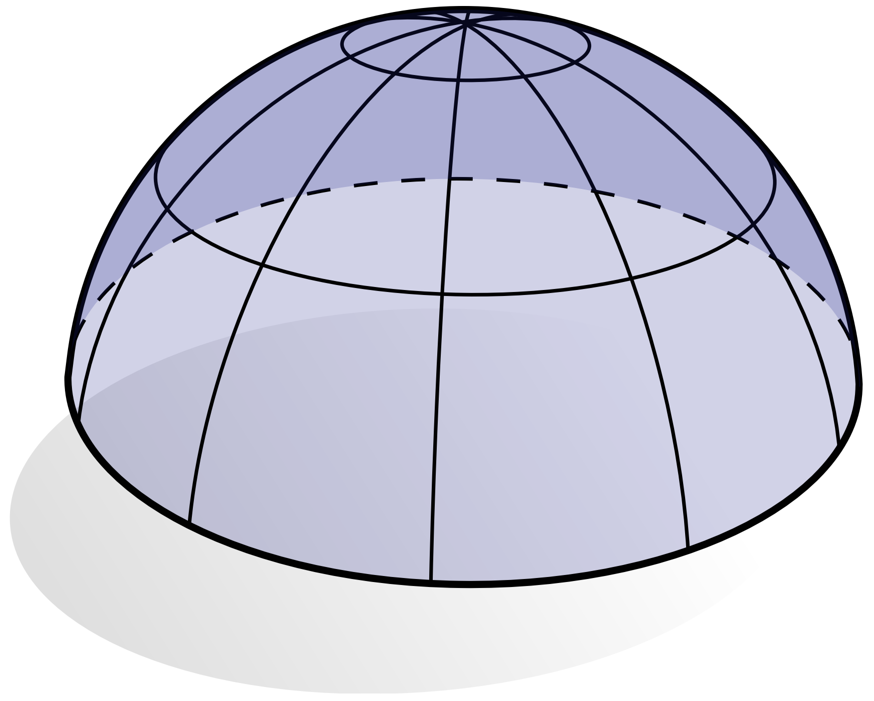
from Knöppel, Crane, Pinkall, Schröder, *"Stripe Patterns on Surfaces"*

Gaussian and Mean Curvature

Gaussian and *mean* curvature also fully describe local bending:

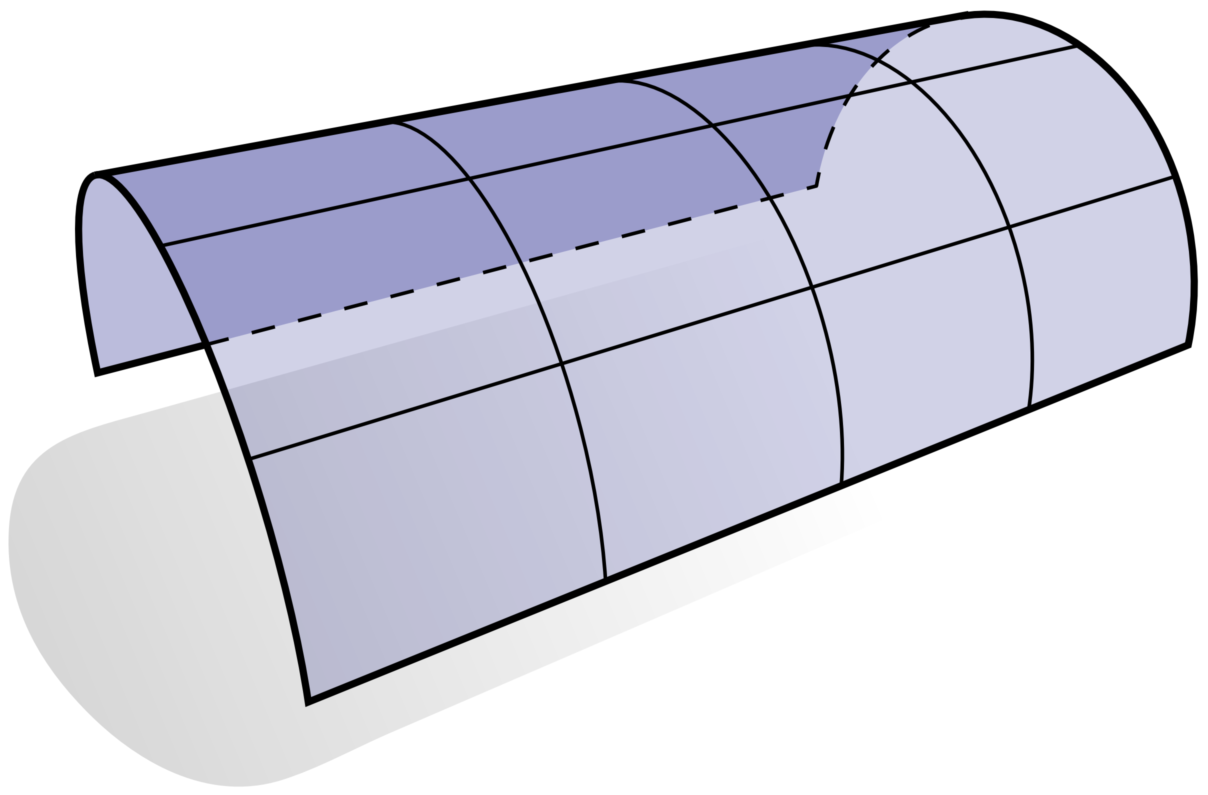
Gaussian $K := \kappa_1 \kappa_2$

mean* $H := \frac{1}{2}(\kappa_1 + \kappa_2)$



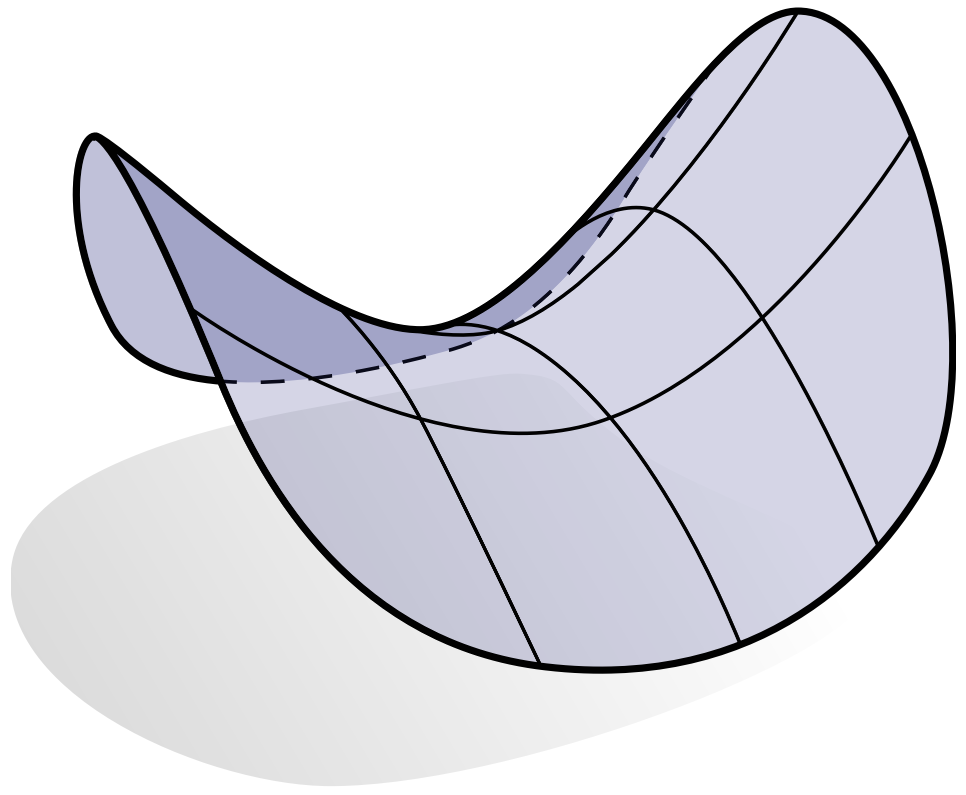
$K > 0$

$H \neq 0$



"developable" $\boxed{K = 0}$

$H \neq 0$



$K < 0$

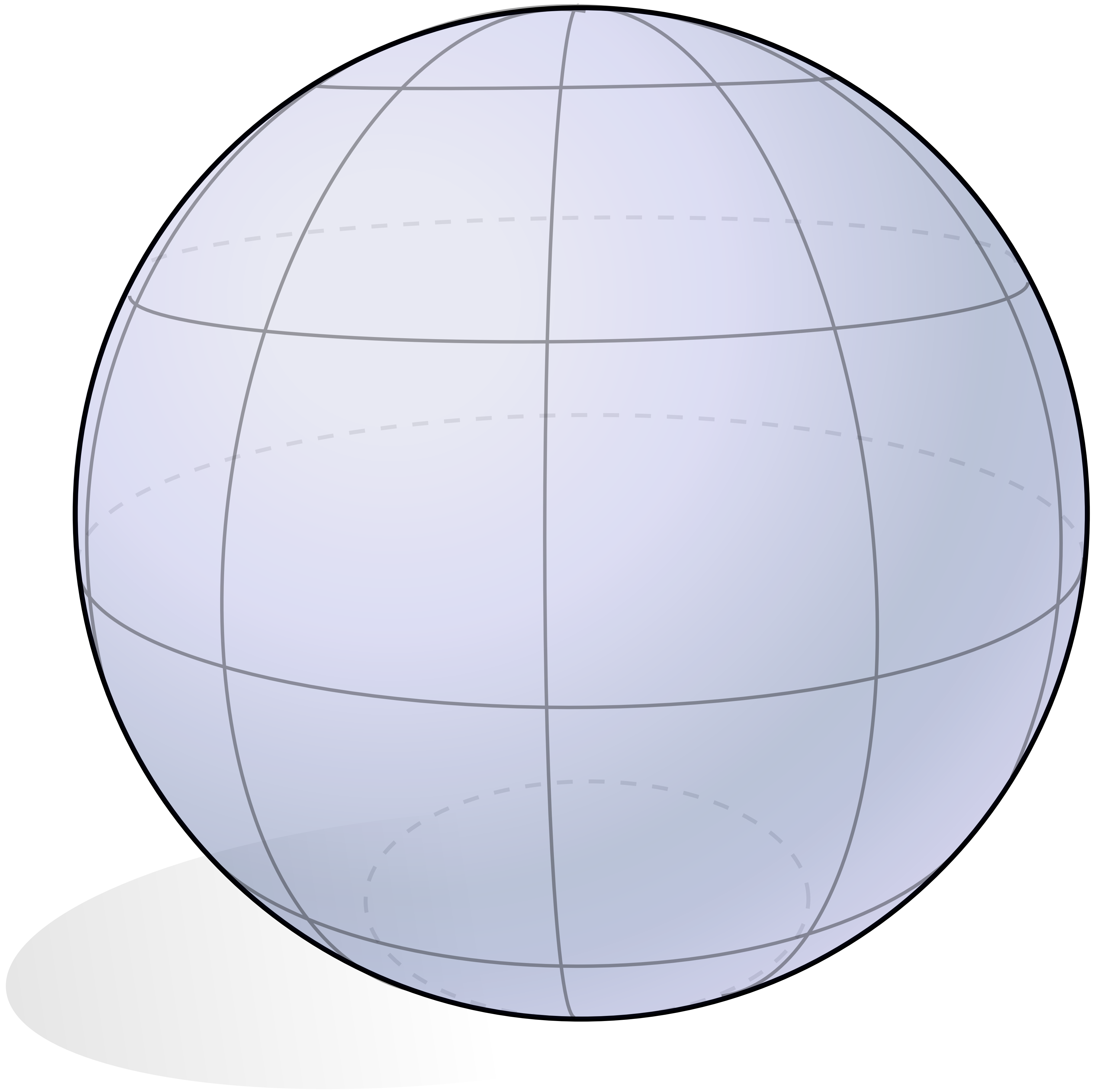
"minimal" $\boxed{H = 0}$

*Warning: another common convention is to omit the factor of $1/2$

Total Mean Curvature?

Theorem (Minkowski): for a regular closed embedded surface,

$$\int_M H\, dA \geq \sqrt{4\pi A}$$

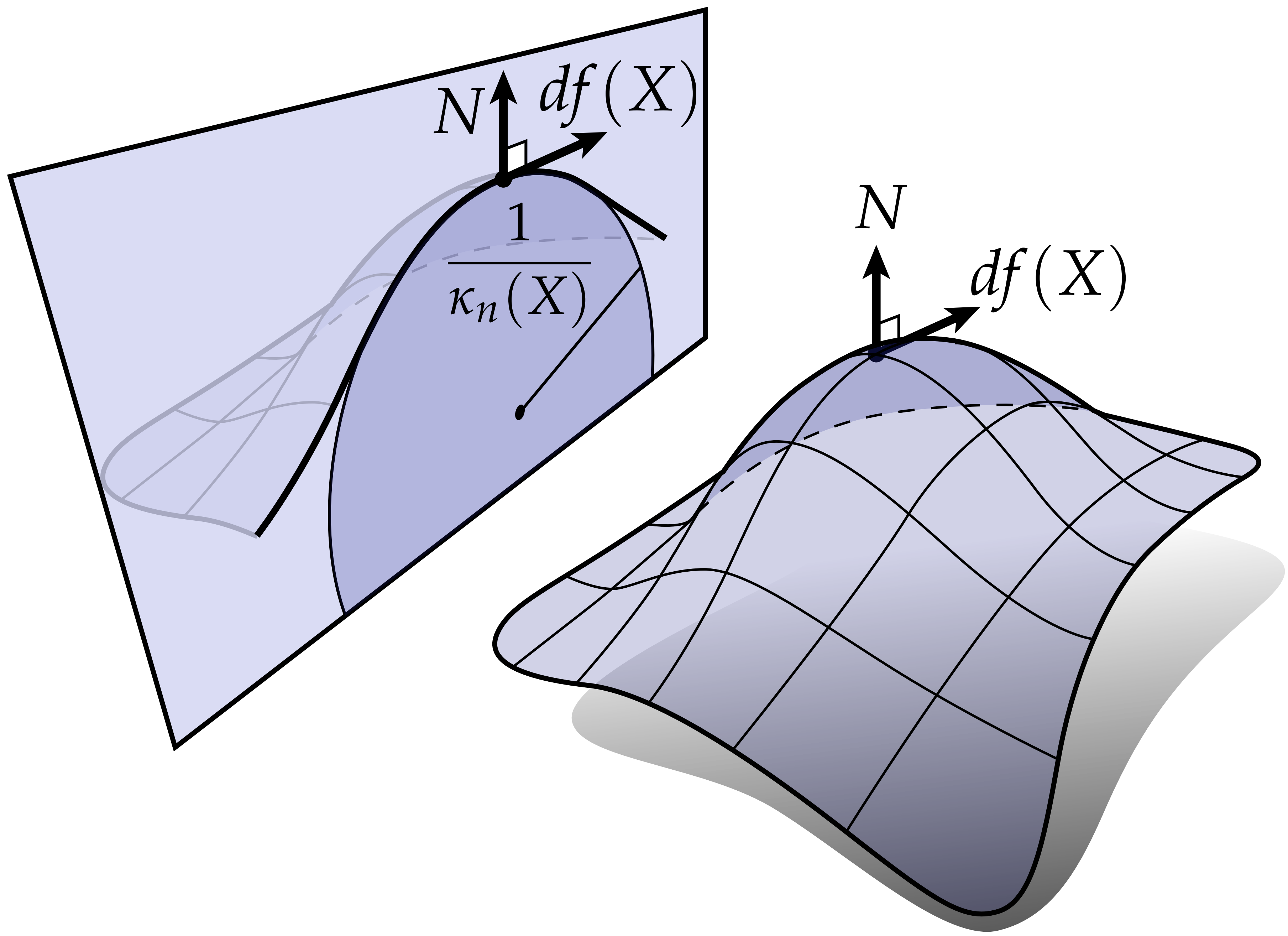


Q: When do we get equality?

A: For a sphere.

Second Fundamental Form

- Second fundamental form is closely related to principal curvature
- Can also be viewed as change in *first* fundamental form under motion in normal direction
- Why "fundamental?"First & second fundamental forms play role in important theorem…



$$\mathbf{II}(X,Y) := \langle dN(X), df(Y) \rangle$$

$$\kappa_N(X) := \frac{df(X), dN(X)}{|df(X)|^2} = \frac{\mathbf{II}(X,X)}{\mathbf{I}(X,X)}$$

Fundamental Theorem of Surfaces

- **Fact.** Two surfaces in R^3 are congruent if and only if they have the same first and second fundamental forms
  - …However, not every pair of bilinear forms **I**, **II** on a domain U describes a valid surface—must satisfy the **Gauss Codazzi** equations
- Analogous to fundamental theorem of plane curves: determined up to rigid motion by curvature
  - …However, for *closed* curves not every curvature function is valid (*e.g.*, must integrate to $2k\pi$)