Lecture 11: High-Dimensional Geometry
Focus of Past High-D Geometry Research

From Low-D to High-D, Change of View
• From calculus view to statistical view
• From infinitesimal analysis to structural/topological analysis

• We will focus on the “odd” behaviors in high-D geometry
Some basic tools

Theorem 2.1 (Markov’s inequality) Let $x$ be a nonnegative random variable. Then for $a > 0$,

$$\text{Prob}(x \geq a) \leq \frac{E(x)}{a}.$$ 

Proof on board

Theorem 2.3 (Chebyshev’s inequality) Let $x$ be a random variable with mean $m$ and variance $\sigma^2$. Then

$$\text{Prob}(|x - m| \geq a\sigma) \leq \frac{1}{a^2}.$$ 

Theorem 2.4 (Law of large numbers) Let $x_1, x_2, \ldots, x_n$ be $n$ samples of a random variable $x$. Then

$$\text{Prob}\left(\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - E(x)\right| > \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}$$
Volume & Surface Distribution
Volume of Unit Sphere Goes to Zero

• Sphere volume:

\[ V(d) = \int_{S^d} \int_{r=0}^{1} r^{d-1} dr d\Omega. \]

At radius \( r \), the surface area of the top of the cone is \( r^{d-1} d\Omega \) since the surface area is \( d-1 \) dimensional and each dimension scales by \( r \).
Volume of Unit Sphere Goes to Zero

• Computation of $A(d)$
  • Trick: compare the integration of $e^{-x^2}$ in Cartesian and Polar systems

Consider a different integral

$$I(d) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2 + x_2^2 + \cdots + x_d^2)} \, dx_1 \cdots dx_d \, dx_1.$$

First, calculate $I(d)$ by integration in Cartesian coordinates.

$$I(d) = \left[ \int_{-\infty}^{\infty} e^{-x^2} \, dx \right]^d = (\sqrt{\pi})^d = \pi^{\frac{d}{2}}.$$

But

$$I(d) = \int_{S^d} d\Omega \int_0^\infty e^{-r^2} r^{d-1} \, dr.$$
Volume of Unit Sphere Goes to Zero

\[ I(d) = \left[ \int_{-\infty}^{\infty} e^{-x^2} \, dx \right]^d = (\sqrt{\pi})^d = \pi^{d/2}. \]

\[ I(d) = \int_{S^d} \int_0^\infty e^{-r^2} r^{d-1} \, dr. \]

\[ \int_0^\infty e^{-r^2} r^{d-1} \, dr = \frac{1}{2} \int_0^\infty e^{-t} t^{d/2} - 1 \, dt = \frac{1}{2} \Gamma \left( \frac{d}{2} \right) \]

and hence, \( I(d) = A(d) \frac{1}{2} \Gamma \left( \frac{d}{2} \right) \)

Lemma 2.5 The surface area \( A(d) \) and the volume \( V(d) \) of a unit-radius sphere in \( d \) dimensions are given by

\[ A(d) = \frac{2\pi^{d/2}}{\Gamma \left( \frac{d}{2} \right)} \quad \text{and} \quad V(d) = \frac{2}{d} \frac{\pi^{d/2}}{\Gamma \left( \frac{d}{2} \right)}. \]
Volume of Unit Sphere Goes to Zero

**Figure 2.5**: Conceptual drawing of a sphere and a cube.
The Volume is Near the Equator

• It turns out that essentially all of the volume of the upper hemisphere lies between the plane $x_1 = 0$ and a parallel plane, $x_1 = \varepsilon$, that is slightly higher.

• For what value of $\varepsilon$ does essentially all the volume lie between $x_1 = 0$ and $x_1 = \varepsilon$? The answer depends on the dimension. For dimension $d$, it is $O\left(\frac{1}{\sqrt{d-1}}\right)$.
The Volume is Near the Equator

\[ \frac{dx_1}{\sqrt{1 - x_1^2}} \text{ radius of } (d - 1)\text{-dimensional sphere} \]
The Volume is Near the Equator

\[
\frac{\text{Volume above slice}}{\text{Volume upper hemisphere}} \leq \frac{\text{Upper bound volume above slice}}{\text{Lower bound volume upper hemisphere}}
\]

Let \( T = \{ \mathbf{x} \mid |\mathbf{x}| \leq 1, x_1 \geq \varepsilon \} \) be the portion of the sphere above the slice.

\[
\text{Volume} (T) = \int_{\varepsilon}^{1} (1 - x_1^2)^{\frac{d-1}{2}} V(d-1) \, dx_1 = V(d-1) \int_{\varepsilon}^{1} (1 - x_1^2)^{\frac{d-1}{2}} \, dx_1.
\]

use the inequality \( 1 + x \leq e^x \) for all real \( x \)

\[
\text{Volume} (T') \leq V(d-1) \int_{\varepsilon}^{\infty} e^{-\frac{d-1}{2} x_1^2} \, dx_1
\]

\[
\leq V(d-1) \int_{\varepsilon}^{\infty} \frac{x_1}{\varepsilon} e^{-\frac{d-1}{2} x_1^2} \, dx_1.
\]

\[
\text{Volume} (T) \leq \frac{1}{\varepsilon(d-1)} e^{-\frac{d-1}{2} \varepsilon^2} V(d-1)
\]
The Volume is Near the Equator

Approximate Volume of upper hemisphere:

\[
\text{volume of the entire upper hemisphere. Clearly, the volume of the upper hemisphere is at least the volume between the slabs } x_1 = 0 \text{ and } x_1 = \frac{1}{\sqrt{d-1}}, \text{ which is at least the volume of the cylinder of radius } \sqrt{1 - \frac{1}{d-1}} \text{ and height } \frac{1}{\sqrt{d-1}}. \text{ The volume of the cylinder is } 1/\sqrt{d-1} \text{ times the } d-1\text{-dimensional volume of the disk } R = \{ x \mid |x| \leq 1; x_1 = \frac{1}{\sqrt{d-1}} \}. \text{ Now } R \text{ is a } d-1\text{-dimensional sphere of radius } \sqrt{1 - \frac{1}{d-1}} \text{ and so its volume is}
\]
\[
\text{Volume}(R) = V(d-1) \left(1 - \frac{1}{d-1}\right)^{(d-1)/2}.
\]

Using \((1 - x)^a \geq 1 - ax\)

\[
\text{Volume}(R) \geq V(d-1) \left(1 - \frac{1}{d-1} \frac{d-1}{2}\right) = \frac{1}{2} V(d-1).
\]

Thus, the volume of the upper hemisphere is at least \(\frac{1}{2\sqrt{d-1}} V(d-1)\).
The Volume is Near the Equator

**Lemma 2.6**  For any $c > 0$, the fraction of the volume of the unit hemisphere above the plane $x_1 = \frac{c}{\sqrt{d-1}}$ is less than $\frac{2}{c} e^{-c^2/2}$.

**Proof:** Substitute $\frac{c}{\sqrt{d-1}}$ for $\varepsilon$ in the above. ■
The Volume is in a Narrow Annulus

The ratio of the volume of a sphere of radius $1 - \varepsilon$ to the volume of a unit sphere in $d$-dimensions is

$$\frac{(1 - \varepsilon)^d V(d)}{V(d)} = (1 - \varepsilon)^d,$$

and thus goes to zero as $d$ goes to infinity when $\varepsilon$ is a fixed constant. In high dimensions, all of the volume of the sphere is concentrated in a narrow annulus at the surface.

Since, $(1 - \varepsilon)^d \leq e^{-\varepsilon d}$, if $\varepsilon = \frac{c}{d}$, for a large constant $c$, all but $e^{-c}$ of the volume of the sphere is contained in a thin annulus of width $c/d$. The important item to remember is that most of the volume of the $d$-dimensional unit sphere is contained in an annulus of width $O(1/d)$ near the boundary. If the sphere is of radius $r$, then for sufficiently large $d$, the volume is contained in an annulus of width $O \left( \frac{r}{d} \right)$. 

Gaussian in High Dimension
Gaussian Distribution

\[ p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} \sigma^d} \exp \left( -\frac{|\mathbf{x}|^2}{2\sigma^2} \right) \]
Expected squared distance of a point from the center of a Gaussian

- A 1-dimensional Gaussian has its mass close to the origin.
Expected squared distance of a point from the center of a Gaussian

- However, as the dimension is increased something different happens
- When $\sigma^2 = 1$, integrating the probability density over a unit sphere centered at the origin yields nearly zero mass since the volume of a unit sphere is negligible.
Expected squared distance of a point from the center of a Gaussian

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- When $\sigma^2 = 1$, integrating the probability density over a unit sphere centered at the origin yields nearly zero mass since the volume of a unit sphere is negligible.
- In fact, one needs to increase the radius of the sphere to $\sqrt{d}$ before there is a significant nonzero volume and hence a nonzero probability mass.
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- When $\sigma^2 = 1$, integrating the probability density over a unit sphere centered at the origin yields nearly zero mass since the volume of a unit sphere is negligible.
- In fact, one needs to increase the radius of the sphere to $\sqrt{d}$ before there is a significant nonzero volume and hence a nonzero probability mass.
- If one increases the radius beyond $\sqrt{d}$, the integral ceases to increase, even though the volume increases, since the probability density is dropping off at a much higher rate. The natural scale for the Gaussian is in units of $\sigma \sqrt{d}$. 
Some Facts

\[
E \left( x_1^2 + x_2^2 + \cdots + x_d^2 \right) = d \ E \left( x_1^2 \right) = d \sigma^2.
\]

The probability mass of a unit-variance Gaussian as a function of the distance from its center is given by \( r^{d-1} e^{-r^2/2} \) times some constant normalization factor where \( r \) is the distance from the center and \( d \) is the dimension of the space. The probability mass function has its maximum at

\[
r = \sqrt{d - 1},
\]
Concentration of Mass for Gaussian

**Theorem 2.11** For a $d$-dimensional, unit variance, spherical Gaussian, for any positive real number $\beta < \sqrt{d}$, all but $3e^{-\frac{\beta^2}{8}}$ of the mass lies within the annulus $\sqrt{d} - \beta \leq r \leq \sqrt{d} + \beta$. 