Lecture 16: Intrinsic Geometry

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Alignment and Registration of Data Sets
Mapping Between Data Sets

- Multiscale mappings
  - Point/pixel level
  - part level

Maps capture what is the same or similar across two data sets.
Why Do We Care About Maps and Alignments?

- To stitch data together
- To transfer information
- To compute distances and similarities
- To perform joint analysis
Extrinsic vs. Intrinsic Alignment

- **Coordinate root mean squared distance**

  \[
  c\text{RMS}^2(P, Q) = \min_{R, t} \frac{1}{n} \sum_{i=1}^{n} \|Rp_i + t - q_i\|^2
  \]

  estimate transform

- **Distance root mean squared distance**

  \[
  d\text{RMS}^2(P, Q) = \frac{1}{n^2} \min_{\sigma} \sum_{i=1}^{n} \sum_{j=1}^{n} (\|p_i - p_j\| - \|q_{\sigma(i)} - q_{\sigma(j)}\|)^2
  \]

  estimate correspondences

  metric space, intrinsic alignment

Gromov-Hausdorff distances
Graph Isomorphism

Intrinsic alignment of manifolds
Why Intrinsic?

Many shapes have natural deformations and articulations that do not change the nature of the shape.

But they change its embedding 3D space.
Why Intrinsic?

Normal distances can change drastically under such deformations. A descriptor based on Euclidean distance histograms, like D2, would fail.
Geodesic / Intrinsic Distances

Near isometric deformations are common for both organic and man-made shapes.

Intrinsic distances are invariant to isometric deformations.

No stretching, shrinking, or tearing.

- Geodesic = intrinsic
- Isometry = length-preserving transform

$M_1$  
$M_2$
We can use geodesic distance histograms.
Geodesic / Intrinsic Distances

Ruggeri et al. 2008
What About **Local** Intrinsic Descriptors?

- Isometrically invariant features
  - Curvature
  - Geodesic Distance
  - Histogram of Geodesic Distances (similar to D2)
  - Global Point Signature
  - Heat Kernel Signature
  - Wave Kernel Signature
Gaussian Curvature

Theorema Egregium ("Remarkable Theorem"): Gaussian curvature is intrinsic.

\[ K = \kappa_1 \kappa_2 \]
Gaussian Curvature

Problems

\[ K = \kappa_1 \kappa_2 \]
Gaussian Curvature

Problems

\[ K = \kappa_1 \kappa_2 \]
Spectral Intrinsic Signatures
Laplace-Beltrami Operator

• Analog of Fourier transform on the sphere, but now on a general 2D manifold
• LB is an operators that can be applied to functions on manifolds to yield other functions

\[ \Delta : C^\infty(M) \to C^\infty(M), \Delta f = \text{div} \nabla f \]

\[ \frac{\partial f}{\partial t} = \Delta f = \text{div} \nabla f \]
LB Eigen-decomposition

- The Laplace-Beltrami operator $\Delta$ has an eigendecomposition

\[ \Delta \phi_i = \lambda_i \phi_i \]

\[
\begin{align*}
\lambda_0 &= 0 \\
\lambda_1 &= 2.6 \\
\lambda_2 &= 3.4 \\
\lambda_3 &= 5.1 \\
\lambda_4 &= 7.6
\end{align*}
\]
Multiscale Basis for a Function Space

\[ f : M \to \mathbb{R} \]

\[ f = \sum_{i=0}^{\infty} a_i \phi_i \]

\[ a_i = \int_{M} f(x) \phi_i(x) d\mu \]
Global Point Signature

\[ GPS(p) = \left( \frac{1}{\sqrt{\lambda_1}} \phi_1(p), \frac{1}{\sqrt{\lambda_2}} \phi_2(p), \frac{1}{\sqrt{\lambda_3}} \phi_3(p), \cdots \right) \]
Global Point Signature

almost invariant under isometries – but not completely canonical

$$GPS(p) = \left(\frac{1}{\sqrt{\lambda_1}}\phi_1(p), \frac{1}{\sqrt{\lambda_2}}\phi_2(p), \frac{1}{\sqrt{\lambda_3}}\phi_3(p), \cdots\right)$$

Diffusion distances are also intrinsic and also canonical

Rustamov et al. 2007
Global Point Signature

\[ GPS(p) = \left( \frac{1}{\sqrt{\lambda_1}} \phi_1(p), \frac{1}{\sqrt{\lambda_2}} \phi_2(p), \frac{1}{\sqrt{\lambda_3}} \phi_3(p), \ldots \right) \]

**Figure 4: Armadillo and its deformations.**

Similar to D2, but use histograms in embedded space (rather than Euclidean)

Rustamov et al. 2007
Global Point Signature

\[ GPS(p) = \left( \frac{1}{\sqrt{\lambda_1}} \phi_1(p), \frac{1}{\sqrt{\lambda_2}} \phi_2(p), \frac{1}{\sqrt{\lambda_3}} \phi_3(p), \ldots \right) \]

- **Pros**
  - Isometry-invariant
  - Global (each point feature depends on entire shape)

- **Cons**
  - Eigenfunctions may flip sign
  - Eigenfunctions might change positions due to deformations
  - Only global

Rustamov et al. 2007
Back to Heat Diffusion

• Heat diffusion on a Riemannian manifold:
  • If $u(x, t)$ is the amount of heat at point $x$ at time $t$, then
  $$\frac{\partial u}{\partial t} = \Delta u$$

• $\Delta$: Laplace-Beltrami Operator (div grad)

• Given an initial distribution $f(x)$ After time $t$

$$f(x, t) = e^{-t\Delta} f$$

$H_t$ heat operator
The Heat Kernel

Heat kernel $k_t(x, y)$:

$$f(x, t) = \int_{\mathcal{M}} k_t(x, y) f(y) dy$$

$k_t(x, y)$: amount of heat transferred from $x$ to $y$ in time $t$. How well $x$ and $y$ are connected at scale $t$.
Background

- Heat Kernel \( k_t(x, y) \). Also the probability density function of Brownian motion on \( \mathcal{M} \):

\[
\mathbb{P} \left( W^t_x \in C \right) = \int_C k_t(x, y) dy
\]

- Intuitively: weighted average over all paths possible between \( x \) and \( y \) in time \( t \)

- Related to Diffusion Distance:

\[
D_t(x, y) = k_t(x, x) - 2k_t(x, y) + k_t(y, y)
\]
- a robust multi-scale measure
- of proximity

Coifman, Lafon
Heat Kernel Properties

Basic Properties

• \( k_t(x, y) = k_t(y, x) \)

• \( k_{t+s}(x, y) = \int_M k_t(x, z) k_s(z, y) dz \)

• \( k_t(x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x) \phi_i(y) \)

Eigenfunctions of LB
Heat Kernel Properties

- Invariant under isometric deformations
  If $T : X \rightarrow Y$ is an isometry, then:
  \[ k_t(X, Y) = k_t(T(x), T(y)) \]

- Conversely: it characterizes the shape up to isometry.
  If $k_t(X, Y) = k_t(T(x), T(y)) \\forall x, y, t$ then:
  \[ T \]
  is an isometry.

This is because:
\[
\lim_{t \downarrow 0} (t \log k_t(x, y)) = -\frac{1}{4} d_M^2(x, y) \\forall x, y
\]
where $d_M(\cdot, \cdot)$ is the geodesic distance
Heat Kernel Properties

**Multiscale:**

For a fixed $x$, as $t$ increases, heat diffuses to larger and larger neighborhoods.

Therefore, $k_t(x, \cdot)$ is determined by (reflects the properties of) a neighborhood that grows with $t$.
Robustness:

$k_t(x, \cdot)$ is the probability density function of BM, a weighted average over all paths, which is generally not very sensitive to local perturbations.

$k_t^M(x, C) = \mathbb{P}(W_t^x \in C)$
Heat Kernel Properties

Robustness:

\[ k_t(x, \cdot) \] is the probability density function of BM, a weighted average over all paths, which is generally not very sensitive to local perturbations.

\[ k_t^{\tilde{M}}(x, C) = \mathbb{P}(\tilde{W}_x^t \in C) \]

Only paths through the modified area \( P \) will change.
Defining a Signature

Let $k_t(x, \cdot)$ be the signature of $x$ at scale $t$

The heat kernel has all the properties we want Except easy comparison …

$k_t(x, \cdot)$ is a function on the entire manifold

Nontrivial to align the domains of such functions across different shapes, or even for different points of the same shape
Let $k_t(x, \cdot)$ be the signature of $x$ at scale $t$

The heat kernel has all the properties we want. Except easy comparison …

We define the Heat Kernel Signature (HKS), by restricting to the diagonal:

$$\text{HKS}(x) = \{k_t(x, x), t \in \mathbb{R}^+\}$$

Now HKSs of two points can be easily compared since they are defined on a common domain (time)

[Sun, Ovsjanikov, G., 2009]
Defining a Signature

Since HKS is a restriction of the heat kernel, it is:
- Robust
- Multiscale

Question: How informative is it?
- Related to Gaussian curvature for small $t$:

$$k_t(x, x) = \frac{1}{4\pi t} \sum_{i=0}^{\infty} a_i t^i \quad a_0 = 1, a_1 = \frac{1}{6}K$$
HKS can be interpreted as a multiscale, robust, intrinsic curvature:
Informative Theorem

The set of all HKSs on a shape almost always defines it up to isometry!

**Theorem:** If $X$ and $Y$ are two compact manifolds, such that $\Delta_X$ and $\Delta_Y$ have only non-repeating eigenvalues, then a homeomorphism $T : X \rightarrow Y$ is an isometry if and only if, for all $x$

$$HKS(x) = HKS(T(x))$$

The set of all HKSs characterizes the intrinsic structure of the manifold
Intuition: Heat kernel is related to the eigenvalues and eigenfunctions of the LB-operator:

\[ \text{HKS}(x, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i^2(x) \]

If eigenvalues do not repeat, we can recover \( \{\lambda_i\} \) and \( \{\phi_i(x)\} \) from \( \text{HKS}(x) \). E.g. \( \lambda_0 = 0 \)

\[ \phi_0^2(x) = \lim_{t \downarrow 0} \text{HKS}(x, t) \]

and

\[ \lambda_1 = \inf \left\{ a \text{ s.t. } \lim_{t \downarrow 0} e^{at} (\text{HKS}(x, t) - \phi_0^2(x)) \neq 0 \right\} \]
Informative Theorem

Nodal domains of the eigenfunctions of LB: domains that are delimited by the zeroes of an eigenfunction

Key property: they are sign interleaved:

No two domains of the same sign can border each other

Note that any mapping that preserves squared values must map a nodal domain to another. Moreover, by fixing a sign of one point, the signs of all other points are fixed by continuity.
Intuition: Heat Kernel is related to the eigenvalues and eigenfunctions of the LB-operator:

$$HKS(x, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i^2(x)$$

After recovering the eigenvalues, and squared eigenfunctions, we know that

$$|\phi_i^Y(T(x))| = |\phi_i^X(x)|$$

We use the properties of nodal domains of eigenfunctions to show:

$$\phi_i^Y(T(x)) = \phi_i^X(x) \quad \text{or} \quad \phi_i^Y(T(x)) = -\phi_i^X(x)$$

Since the eigenvalues + eigenfunctions define the manifold, the theorem follows
Informative Theorem

How general is the theorem?

If there are repeated eigenvalues, it does not hold:

On the sphere, \( \text{HKS}(x) = \text{HKS}(y) \ \forall \ x, y \) but there are non-isometric maps between spheres.

Uhlenbeck’s Theorem (1976): for “almost any” metric on a 2-manifold \( X \), the eigenvalues of \( \Delta_X \) are non-repeating.
Conclusion: HKS is informative for individual points and, as a set, for the entire shape.

Can be used both for multiscale point matching and for shape comparison.

\[
\text{HKS}(x) = \{k_t(x, x), t \in \mathbb{R}^+\}
\]
Applications

- Multi-scale matching with HKS, structure discovery
- Shape comparison
- Shape retrieval using HKS
- Spectral version of Gromov-Hausdorff
Comparing points through their HKS signatures:
Multiscale Matching

Comparing points through their HKS signatures:

Medium scale

Full scale
Finding similar points – robustly:

Medium scale

Full scale
Finding similar points across multiple shapes:

Medium scale

Full scale
The End